Modal Logic

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A graduate class, TT17

Resources

Textbook (weeks 1–4): *Logic for Philosophy*, Ted Sider (OUP)

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I. Brief review of propositional logic (PL)

I.1. Official Syntax (LfP 2.1)

The language of PL has the following primitive vocabulary:

- Connectives: $\sim$, $\rightarrow$
- Infinitely many sentence letters: $P, Q, R, \ldots$ (with or without numerical subscripts)
- Parentheses: (,)

Well-formed formulas (wffs, alias: formulas, sentences) are defined as follows:

Definition I.1.1 (PL-wff, LfP 26)**.**

- If α is a sentence letter, α is a PL-wff
- If ϕ and ψ are PL-wffs, then $\sim\phi$ and $(\phi \rightarrow \psi)$ are also PL-wffs
- Only strings that can be shown to be PL-wffs using the above clauses are PL-wffs

Worked Example A.

According to the letter of the definition, which of the following strings are PL-wffs?

Q	$\sim(\phi \rightarrow \psi)$	$\sim\sim\sim\sim P$
$P \rightarrow Q$	$\sim(\sim R \rightarrow Q_4)$	$(P \wedge \sim Q)$
$(P_5 \rightarrow Q_{17})$	$\sim(\sim P)$	$\sim((\sim P \rightarrow Q) \rightarrow R)$

I.2. Unofficial Syntax

Unless we're specifically concerned with syntactic matters, we'll usually permit ourselves to be looser about syntax.

Unofficial connectives (LfP 27)

To make writing down formulas easier we help ourselves to the following abbreviations in the metalanguage:

- $(\phi \wedge \psi)$ is short for $\sim(\phi \rightarrow \sim\psi)$
- $(\phi \vee \psi)$ is short for $(\sim\phi \rightarrow \psi)$
- $(\phi \leftrightarrow \psi)$ is short for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$

Note. Here, following Sider, we differ from the *Logic Manual*, which takes all five connectives as primitive.

Remark. Working with a short list of official connectives tends to make theory harder but metatheory easier.

Bracketing conventions

We'll apply standard bracketing conventions, mostly omitting outer brackets. (See e.g. the *Logic Manual*.)

Worked Example B.

Write down the following in primitive notation:

$$\sim(P \vee Q) \qquad P \wedge \sim Q \qquad (P \leftrightarrow Q)$$

I.3. Semantics: interpretations and valuations (LfP 2.3)

Definition I.3.1 (LfP 29)**.** A *PL-interpretation* is a function $\mathscr{I}$ that assigns each sentence letter a truth value, either 1 ('true') or 0 ('false').

Definition I.3.2 (LfP 30)**.** For any PL-interpretation $\mathscr{I}$, the *PL-valuation for* $\mathscr{I}$—symbolised $V_{\mathscr{I}}$—is the (unique) function that assigns 1 or 0 to each wff as follows:

- $V_{\mathscr{I}}(\alpha) = \mathscr{I}(\alpha)$, for each sentence letter α
- $V_{\mathscr{I}}(\phi \to \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 0$ or $V_{\mathscr{I}}(\psi) = 1$
- $V_{\mathscr{I}}(\sim\phi) = 1$ iff $V_{\mathscr{I}}(\phi) = 0$

Remark. I'll often read "$V_{\mathscr{I}}(\phi) = 1$" as "$\phi$ is true in $\mathscr{I}$". Given their definitions, this secures the expected truth-conditions for our unofficial connectives:

- $V_{\mathscr{I}}(\phi \vee \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 1$ or $V_{\mathscr{I}}(\psi) = 1$
- $V_{\mathscr{I}}(\phi \wedge \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = 1$ and $V_{\mathscr{I}}(\psi) = 1$
- $V_{\mathscr{I}}(\phi \leftrightarrow \psi) = 1$ iff $V_{\mathscr{I}}(\phi) = V_{\mathscr{I}}(\psi)$

I.4. Semantics: validity and consequence

Definition I.4.1 (LfP 34)**.**

- A sentence ϕ is *PL-valid* iff for every interpretation $\mathscr{I}$, $V_{\mathscr{I}}(\phi) = 1$.
- A sentence ϕ is a *semantic consequence* of a set of sentences Γ iff $V_{\mathscr{I}}(\phi) = 1$ for every interpretation $\mathscr{I}$ such that $V_{\mathscr{I}}(\gamma) = 1$ for each $\gamma \in \Gamma$.

Warning. Even in logic, terminology varies from source to source:

- When ϕ is valid, ϕ is also sometimes called a tautology or a logical truth (and, in symbols, we write $\vDash_{\mathrm{PL}} \phi$).

  Note that, unlike some authors, Sider reserves 'valid' for formulas (not arguments).
- Semantic consequence is also frequently called 'entailment' (symbolised $\Gamma \vDash_{\mathrm{PL}} \phi$). Although some authors use entailment for mere necessary implication.

Remark. Semantic consequence can be re-characterised in terms of 'satisfaction'. Say that $\mathscr{I}$ satisfies:

- a sentence ϕ if $V_{\mathscr{I}}(\phi) = 1$
- a set Γ if $V_{\mathscr{I}}(\gamma) = 1$ for each $\gamma \in \Gamma$.

Then $\Gamma \vDash \phi$ iff every interpretation that satisfies Γ also satisfies ϕ.

I.5. Establishing validity (LfP 2.4)

Truth-tables

Recall that we can establish validity and non-validity using truth-tables.

> **Worked Example C.** Use truth tables to demonstrate the following:
>
> $$\text{(i)} \vDash (\phi \to (\psi \to \phi)) \quad \text{(ii)} \sim\psi, \phi \to \psi \vDash \sim\phi$$

Informal semantic arguments

Truth-tables are fine for PL. But they won't work when we come to modal logic. So let's introduce another means for establishing validity.

- To show $\vDash \phi$, give a semantic argument to show that the supposition that $\mathrm{V}_{\mathscr{I}}(\phi) = 0$ leads to a contradiction (using the truth clauses in the definition of a valuation)
- To show $\phi_1, \ldots, \phi_n \vDash \psi$, give a semantic argument to show that the suppositions that $\mathrm{V}_{\mathscr{I}}(\phi_1) = \cdots = \mathrm{V}_{\mathscr{I}}(\phi_n) = 1$ and $\mathrm{V}_{\mathscr{I}}(\psi) = 0$ jointly lead to a contradiction

> **Worked Example D.** Give informal semantic arguments to demonstrate the following:
>
> $$\text{(i)} \vDash (\phi \to (\psi \to \phi)) \quad \text{(ii)} \sim\psi, \phi \to \psi \vDash \sim\phi$$

See also LfP, Examples 2.2 and 2.3, for further worked examples.

> **Exercise 1.** Give an informal semantic argument to show that:
>
> $$\vDash (\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$$
>
> (Don't use truth tables.)

(Proof theory: deferred until week 2!)

II. Towards modal propositional logic (MPL)

II.1. Expressive strength and weakness in PL (cf. LfP 3.1)

Definition II.1.1. An *n-ary truth function* is a function that maps each n-tuple of truth values to a truth value (and is otherwise undefined).

Remark. An n-ary truth function is uniquely represented by the corresponding truth table

Example. For example the truth tables below characterise binary truth functions $f_\wedge$ and $f_\leftarrow$

t_1	t_2	$f_\wedge(t_1, t_2)$	$f_\leftarrow(t_1, t_2)$
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	1

Definition II.1.2 (LfP 68)**.** Let f be an n-ary truth function. Let $\phi(P_1, \ldots, P_n)$ be a PL-wff which contains $P_1, \ldots, P_n$ as its only sentence letters. Then $\phi(P_1, \ldots, P_n)$ *symbolizes* (or *expresses*) f iff, for each PL-interpretation $\mathscr{I}$:

$$\mathrm{V}_{\mathscr{I}}(\phi(P_1 \ldots, P_n)) = f(\mathscr{I}(P_1), \ldots, \mathscr{I}(P_n))$$

Remark. In other words, $\phi(P_1, \ldots, P_n)$ symbolises a truth function f if they have the same truth table. e.g. $P_1 \wedge P_2$ and $P_2 \rightarrow P_1$ symbolise $f_\wedge$ and $f_\leftarrow$.

Fact II.1.3. Every n-ary truth function is symbolised by a PL-sentence $\phi(P_1, \ldots, P_n)$.

Fact II.1.4. Every PL-sentence $\phi(P_1, \ldots, P_n)$ symbolises an n-ary truth function.

Remark. Fact 1 demonstrates that PL has maximal expressive strength when it comes to symbolising truth-functions. But Fact 2 shows it goes no further.

This shows that we cannot adequately capture non-truth-functional English connectives such as ‘Tim knows that P’ or ‘It could be the case that P’.

To capture these, we need connectives whose semantic contribution cannot be summarised in a truth table.

II.2. Syntax (LfP 6.1)

The syntax is just like PL except that we add a new unary connective $\Box$ (read: 'box' or 'it is necessary that') which functions syntactically just like negation.

Official Syntax

The language of PL has the following primitive vocabulary:

- Connectives: $\sim$, $\rightarrow$, $\Box$
- Infinitely many sentence letters: $P, Q, R, \ldots$ (with or without numerical subscripts)
- Parentheses: (,)

Definition II.2.1 (MPL-wff, LfP 135)**.**

- If α is a sentence letter, α is a MPL-wff
- If ϕ and ψ are MPL-wffs, then $\sim\phi$, $(\phi \rightarrow \psi)$ and $\Box\phi$ are also MPL-wffs
- Only strings that can be shown to be MPL-wffs using the above clauses are MPL-wffs

Unofficial connective

- $\Diamond\phi$ is short $\sim\Box\sim\phi$. ($\Diamond$ may be read 'diamond' or 'possibly'.)

II.3. SMPL-semantics: models

Let's start with a simplified version of MPL: SMPL. (We'll come to the full MPL shortly.)

Definition II.3.1. A *simplified MPL-model* (*SMPL-model*) is a pair: $\langle \mathscr{W}, \mathscr{I} \rangle$ where:

- $\mathscr{W}$ is a non-empty set ("the set of possible worlds")
- $\mathscr{I}$ is a function that assigns each sentence-letter–world pair a truth value, 1 or 0 ("interpretation function")

Example (A toy model)**.** $\mathscr{W} = \{0, 1, 2\}$

$\mathscr{I}(P, 0) = 1$	$\mathscr{I}(Q, 0) = 0$
$\mathscr{I}(P, 1) = 1$	$\mathscr{I}(Q, 1) = 1$
$\mathscr{I}(P, 2) = 0$	$\mathscr{I}(Q, 2) = 0$

II.4. SMPL-semantics: valuations

Definition II.4.1. Given an SMPL-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{I} \rangle$, the *valuation for* $\mathscr{M}$, $\mathrm{V}_{\mathscr{M}}$, is the two place function that assigns 0 or 1 to each MPL-wff, for each $w \in \mathscr{W}$, as follows:

- $\mathrm{V}_{\mathscr{M}}(\alpha, w) = \mathscr{I}(\alpha, w)$, for each sentence letter α
- $\mathrm{V}_{\mathscr{M}}(\phi \to \psi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ or $\mathrm{V}_{\mathscr{M}}(\psi, w) = 1$
- $\mathrm{V}_{\mathscr{M}}(\sim\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$
- $\mathrm{V}_{\mathscr{M}}(\Box\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, v) = 1$ for all $v \in \mathscr{W}$

Remark. This generates the expected truth-conditions for diamond:

- $\mathrm{V}_{\mathscr{M}}(\Diamond\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, v) = 1$ for some $v \in \mathscr{W}$

Worked Example E. Let $\mathscr{M}$ be the toy model in the above example. Compute:

$$\begin{array}{ll} \mathrm{V}_{\mathscr{I}}(\Box P, 0) & \mathrm{V}_{\mathscr{I}}(\Box P, 2) \\ \mathrm{V}_{\mathscr{I}}(\Diamond\sim P, 0) & \mathrm{V}_{\mathscr{I}}(\Diamond Q \leftrightarrow \Box P, 0) \\ \mathrm{V}_{\mathscr{I}}(\Diamond P, 2) & \mathrm{V}_{\mathscr{I}}(\Diamond\Diamond\sim P, 0) \end{array}$$

II.5. Extension and intension

Let $\mathscr{M} = \langle \mathscr{W}, \mathscr{I} \rangle$ be an SMPL model and ϕ be a MPL-wff.

Definition II.5.1. Call the truth-value $\mathrm{V}_{\mathscr{M}}(\phi, w)$ the *extension* of ϕ in w (relative to $\mathscr{M}$).

Fact II.5.2. The PL-connectives are extensional: i.e. the extension of a complex PL-wff is a function of the extensions of its immediate constituents, e.g.:

- $\mathrm{V}_{\mathscr{M}}(\sim\phi, w) = f_{\sim}(\mathrm{V}_{\mathscr{M}}(\phi, w)) \quad (= 1 - \mathrm{V}_{\mathscr{M}}(\phi, w))$
- $\mathrm{V}_{\mathscr{M}}(\phi \wedge \psi, w) = f_{\wedge}(\mathrm{V}_{\mathscr{M}}(\phi, w), \mathrm{V}_{\mathscr{M}}(\psi, w)) \quad (= \mathrm{V}_{\mathscr{M}}(\phi, w) \cdot \mathrm{V}_{\mathscr{M}}(\psi, w))$

But $\Box$ is not extensional: i.e. there is *no* function f such that, for any $\mathscr{M}$:

- $\mathrm{V}_{\mathscr{M}}(\Box\phi, w) = f(\mathrm{V}_{\mathscr{M}}(\phi, w))$

However $\Box$ is 'intensional' in a natural sense.

Definition II.5.3. The intension of ϕ (relative to $\mathscr{M}$)—written $[\phi]_{\mathscr{M}}$—may be defined:

$$[\phi]_{\mathscr{M}} = \{w : \mathrm{V}_{\mathscr{M}}(\phi, w) = 1\}$$

Fact II.5.4. The intension of a complex MPL-wff is a function of the intensions of its immediate constituents, e.g.:

$$[\sim\phi]_{\mathscr{M}} = W - [\phi]_{\mathscr{M}} \qquad [\phi \wedge \psi]_{\mathscr{M}} = [\phi]_{\mathscr{M}} \cap [\psi]_{\mathscr{M}} \qquad [\Box\phi]_{\mathscr{M}} = \begin{cases} W \text{ if } [\phi]_{\mathscr{M}} = W \\ \varnothing \text{ otherwise} \end{cases}$$

II.6. SMPL-semantics: validity

Definition II.6.1 (Validity). Given an MPL-wff ϕ:

- ϕ is valid in an SMPL-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{I} \rangle$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 1$ for every $w \in \mathscr{W}$
- ϕ is SMPL-valid if ϕ is valid in every SMPL-model.

Remark. In other words, ϕ is SMPL-valid if true at every world of every SMPL-model.

When this is so, we write $\vDash_{\text{SMPL}} \phi$.

II.7. Establishing validity

To establish SMPL-validity we can employ informal semantic arguments akin to those used to establish PL-validity above.

To show $\vDash_{\text{SMPL}} \phi$ it suffices to show that the supposition that $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ leads to a contradiction (for $\mathscr{M} = \langle \mathscr{W}, \mathscr{I} \rangle$ and $w \in \mathscr{W}$).

Worked Example F. Show $\vDash_{\text{SMPL}} \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$

Exercise 2. Give informal arguments to demonstrate the following:

(a) $\vDash_{\text{SMPL}} \Box(P \wedge Q) \rightarrow \Box P \wedge \Box Q$

(b) $\vDash_{\text{SMPL}} \Box P \rightarrow \Box\Box P$

(c) $\vDash_{\text{SMPL}} \Diamond P \rightarrow \Box\Diamond P$

II.8. Establishing invalidity

To establish the SMPL-invalidity of ϕ we need to specify a countermodel—i.e. an SMPL-model $\langle \mathscr{W}, \mathscr{I} \rangle$ such that $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ for some $w \in \mathscr{W}$.

Worked Example G. Show $\nvDash_{\text{SMPL}} {\sim}P \rightarrow {\sim}\Diamond P$

Exercise 3. Specify countermodels that establish the following:

$$\nvDash_{\text{SMPL}} \Box\Diamond P \qquad \nvDash_{\text{SMPL}} \Diamond(P \wedge \Box Q) \rightarrow \Box(P \wedge Q)$$

III. Modal Propositional Logic (MPL)

III.1. Motivating MPL: notable SMPL validities

The SMPL-semantics validates the following modal schemas:[1]

(D) $\vDash_{\text{SMPL}} \Box\phi \rightarrow \Diamond\phi$

(T) $\vDash_{\text{SMPL}} \Box\phi \rightarrow \phi$

(B) $\vDash_{\text{SMPL}} \Diamond\Box\phi \rightarrow \phi$

(4) $\vDash_{\text{SMPL}} \Box\phi \rightarrow \Box\Box\phi$

(5) $\vDash_{\text{SMPL}} \Diamond\Box\phi \rightarrow \Box\phi$

But are these formulas intuitively valid? It depends on how we understand $\Box$.

Worked Example H. Are (D), (T), (B), (4), and (5) are intuitively valid when $\Box$ and $\Diamond$ are read as below?

(a) $\Box\phi$: 'It will be the case that ϕ at every future time'

$\Diamond\phi$: 'It will be the case that ϕ at some future time'

(b) $\Box\phi$: 'You are required to make it the case that ϕ'

$\Diamond\phi$: 'You are permitted to make it the case that ϕ'

Notation. These, and some other, readings of $\Box$ and $\Diamond$ have canonical notations:

$G\phi$: 'It will [is *going* to] be the case that ϕ at all future times'

$F\phi$: 'It will be the case that ϕ at some *future* times'

$H\phi$: 'It *has* been the case that ϕ at all past times'

$P\phi$: 'It was the case that ϕ at some *past* times'

III.2. Motivating MPL: accessibility

What's gone wrong e.g. in the temporal case?

- The obvious culprit is the SMPL-truth-conditions for $\Box$. For $\Box = G$, we get:

  SMPL: $G\phi$ is true at t iff ϕ is true at every time t'

- But intuitively, the correct truth-condition is this:

  MPL: $G\phi$ is true at t iff ϕ is true at every time t' later than t.

[1] A schema is said to be valid if each instance of it is valid. Compare LfP, 2.4.1

III.3. MPL-semantics: models (LfP 6.3)

MPL-models add an 'accessibility relation' to SMPL-models:

Definition III.3.1 (LfP 139)**.** An *MPL-model* is a triple: $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ where:

- $\mathscr{W}$ is a non-empty set ("the set of possible worlds")
- $\mathscr{R}$ is a binary relation over $\mathscr{W}$ ("accessibility relation")
- $\mathscr{I}$ is a two-place function that assigns each sentence-letter–world pair a truth-value, 1 or 0 ("interpretation function")

Remarks.

- $\mathscr{W}$ and $\mathscr{I}$ are the same as in the definition of SMPL-model.
- $\mathscr{R}wv$ is read 'v is accessible from w' or 'v is possible relative to w' (informally: 'w sees v', etc.)

III.4. MPL-semantics: valuations

Definition III.4.1 (LfP 139–40)**.** Given an MPL-model $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$, the *valuation for* $\mathscr{M}$, $\mathrm{V}_{\mathscr{M}}$, is the function that assigns 0 or 1 to each MPL-wff for each $w \in \mathscr{W}$ as follows:

- $\mathrm{V}_{\mathscr{M}}(\alpha, w) = \mathscr{I}(\alpha, w)$, for each sentence letter α
- $\mathrm{V}_{\mathscr{M}}(\phi \rightarrow \psi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ or $\mathrm{V}_{\mathscr{M}}(\psi, w) = 1$
- $\mathrm{V}_{\mathscr{M}}({\sim}\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$
- $\mathrm{V}_{\mathscr{M}}(\Box\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, v) = 1$ for all $v \in \mathscr{W}$ such that $\mathscr{R}wv$

Remark. The only change to the SMPL-semantic clauses is the switch from "truth at all worlds" to "truth at all accessible worlds" in the final clause.

We continue to read '$\mathrm{V}_{\mathscr{M}}(\phi, w)$' as '$\phi$ is true in w (in $\mathscr{M}$)'.

Remark. This generates the following truth-conditions for diamond:

- $\mathrm{V}_{\mathscr{M}}(\Diamond\phi, w) = 1$ iff $\mathrm{V}_{\mathscr{M}}(\phi, v) = 1$ for some $v \in \mathscr{W}$ such that $\mathscr{R}wv$

Example (A toy model)**.** $\mathscr{W} = \{0, 1, 2\}$; $\mathscr{R} = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 2\rangle\}$.

$$\begin{array}{ll} \mathscr{I}(P, 0) = 1 & \mathscr{I}(Q, 0) = 0 \\ \mathscr{I}(P, 1) = 1 & \mathscr{I}(Q, 1) = 1 \\ \mathscr{I}(P, 2) = 0 & \mathscr{I}(Q, 2) = 0 \end{array}$$

Worked Example I. Let $\mathscr{M}$ be the toy model in the above example. Compute:

$$\begin{array}{ll} \mathrm{V}_{\mathscr{I}}(\Box P, 0) & \mathrm{V}_{\mathscr{I}}(\Box P, 2) \\ \mathrm{V}_{\mathscr{I}}(\Diamond{\sim}P, 0) & \mathrm{V}_{\mathscr{I}}(\Diamond Q \leftrightarrow \Box P, 0) \\ \mathrm{V}_{\mathscr{I}}(\Diamond P, 2) & \mathrm{V}_{\mathscr{I}}(\Diamond\Diamond{\sim}P, 0) \end{array}$$

III.5. MPL semantics: validity and modal systems

Modal systems

Different modal systems result from imposing different conditions on accessibility:

System	Condition(s) on $\mathscr{R}$	i.e.
K	—	
D	$\mathscr{R}$ is serial on $\mathscr{W}$	for each $w \in \mathscr{W}$, there is some u s.t. $\mathscr{R}wu$
T	$\mathscr{R}$ is reflexive on $\mathscr{W}$	for each $w \in \mathscr{W}$, $\mathscr{R}ww$
B	$\mathscr{R}$ is reflexive on $\mathscr{W}$	
	$\mathscr{R}$ is symmetric	for each w, v, $\mathscr{R}wv$ implies $\mathscr{R}vw$
S4	$\mathscr{R}$ is reflexive on $\mathscr{W}$	
	$\mathscr{R}$ is transitive	for each w, v, u, $\mathscr{R}wv$ and $\mathscr{R}vu$ jointly imply $\mathscr{R}wu$
S5	$\mathscr{R}$ is reflexive on $\mathscr{W}$	
	$\mathscr{R}$ is symmetric	
	$\mathscr{R}$ is transitive	

Definition III.5.1 (Valid in a model, LfP 141)**.** Let $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ be an MPL-model, ϕ an MPL-wff:

- ϕ is *valid in* $\mathscr{M}$ iff $\mathrm{V}_{\mathscr{M}}(\phi, w) = 1$ for every $w \in \mathscr{W}$.

Definition III.5.2 (S-valid, LfP 141)**.** Let S be one of K, D, T, B, S4 or S5. Let ϕ be an MPL-wff:

- ϕ is *valid in S* iff ϕ is valid in every S-model.

Remark. In other words, ϕ is S-valid if true at every world of every S-model.

When this is so, we write $\vDash_{\mathrm{S}} \phi$.

III.6. Establishing validity (LfP 6.3.2)

To establish MPL-validity we can employ informal semantic arguments akin to those used to establish SMPL-validity above.

To show $\vDash_S \phi$ it suffices to show that the supposition that $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ leads to a contradiction given the condition on $\mathscr{R}$ imposed by S (for $\mathscr{M} = \langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ and $w \in \mathscr{W}$).

Worked Example J. Give informal semantic arguments to demonstrate the following:

(D) $\vDash_{\mathrm{D}} \Box\phi \to \Diamond\phi$

(B) $\vDash_{\mathrm{B}} \phi \to \Box\Diamond\phi$

Exercise 4. Give informal semantic arguments to demonstrate the following:

(T) $\vDash_{\mathrm{T}} \Box\phi \to \phi$

(4) $\vDash_{\mathrm{S4}} \Box\phi \to \Box\Box\phi$

(5) $\vDash_{\mathrm{S5}} \Diamond\phi \to \Box\Diamond\phi$

III.7. Establishing invalidity (LfP 6.3.3)

To establish the S-invalidity of ϕ we need to specify a countermodel—i.e. an S-model $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ such that $\mathrm{V}_{\mathscr{M}}(\phi, w) = 0$ for some $w \in \mathscr{W}$.

Worked Example K. Specify countermodels to demonstrate the following:

(D) $\nvDash_{\mathrm{K}} \Box\phi \to \Diamond\phi$

(B) $\nvDash_{4} \phi \to \Box\Diamond\phi$

Exercise 5. Specify countermodels to demonstrate the following:

(T) $\nvDash_{\mathrm{K}} \Box\phi \to \phi$

(4) $\nvDash_{\mathrm{B}} \Box\phi \to \Box\Box\phi$

(5) $\nvDash_{4} \Diamond\phi \to \Box\Diamond\phi$

$\nvDash_{\mathrm{B}} \Diamond\phi \to \Box\Diamond\phi$