Modal Logic

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III.8 Proof theory for K (Compare LfP 6.4)

We'll present an axiomatic proof system for K.

Definition III.8.1 (MPL-tautology, LfP 160)**.** An MPL-formula ϕ is said to be an *MPL-tautology* if ϕ results from a PL-tautology by uniform substitution of MPL-wffs for sentence letters.

Remark. See LfP 102 for a list of some well-known PL-tautologies.

Worked Example A. Which of the following are propositional tautologies?

$$(P \wedge {\sim}P) \to \Box P \qquad \Box P \to \Diamond P \qquad {\sim}\Box P \leftrightarrow \Diamond{\sim}P \qquad \Box{\sim}P \leftrightarrow {\sim}\Diamond P$$

Warning. Sider's term may be misleading. Note that not all K-valid MPL-formulas are MPL-tautologies (although the converse holds).

Axioms and Rules for K.

- *Axioms*: All MPL-tautologies are K-axioms, plus all instances of the following:

  (K) $\quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

- *Rules*: All instances of the following are K-rules:

  MP $\dfrac{\phi \qquad \phi \to \psi}{\psi}$ $\qquad$ **Nec** $\dfrac{\phi}{\Box\phi}$

Definition of an S-theorem A wff ϕ is *provable in S*, or an *S-theorem*, (in symbols: $\vdash_S \phi$) if there is a finite sequence of wffs:

$$\begin{array}{c}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{array}$$

where the last line, ϕ_n, is ϕ and for each line, ϕ_i $(i = 1, \ldots, n)$, either:

- ϕ_i is an S-axiom, or
- ϕ_i follows from earlier wffs in the sequence via an S-rule:

$$\frac{\phi_{j_1} \ldots \phi_{j_n}}{\phi_i}$$

with $j_1, \ldots, j_n < i$.

Worked Example B. Construct proofs to show the following:

(i) $\vdash_K \Box(P \to P)$

(ii) $\vdash_K \Box(P \to Q) \to \Box({\sim}Q \to {\sim}P)$

Remark. Even when we're doing proof theory, we almost never write out full axiomatic proofs. Instead we convince ourselves that such a proof exists, telescoping steps using derived rules of the form:

$$\frac{\phi_1, \ldots, \phi_n}{\psi}$$

Such a rule is said to be *S-admissible* if the S-provability of ϕ_1,..., ϕ_n implies the S-provability of ψ.

A derived rule. Suppose $\phi_1, \ldots, \phi_n \vDash_{PL} \psi$. Then the following rule is K-admissible:

$$\textsf{PL}\ \frac{\phi_1, \ldots, \phi_n}{\psi}$$

More generally, PL is K-admissible whenever $\phi_1 \to (\phi_2 \to (\cdots \to (\phi_n \to \psi) \cdot\cdot))$ is an MPL-tautology.

Another derived rule. Let O be $\Box$ or $\Diamond$. Then the following is K-admissible:

$$\textsf{Becker}\ \frac{\phi \to \psi}{O\phi \to O\psi}$$

Worked Example C. Give abbreviated proofs to demonstrate the following:

(i) $\vdash_{\mathrm{K}} \Box(P \to Q) \to (\Diamond P \to \Diamond Q)$

(ii) $\vdash_{\mathrm{K}} (\Box P \wedge \Diamond Q) \to \Diamond(P \wedge Q)$

Extended Remark. Our axiomatic system differs from Sider's. We've trivialized the PL-part of the proof system by admitting all substitution instances of PL-valid formulas as axioms. Sider instead uses the following axioms for PL

(PL1) $\phi \to (\psi \to \phi)$

(PL2) $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

(PL3) $({\sim}\psi \to {\sim}\phi) \to (({\sim}\psi \to \phi) \to \psi)$

- This makes official proofs in Sider's system longer than in ours.

  e.g. the official proof of worked example B (ii) above is (much) longer in Sider's system. We first have to establish $(P \to Q) \to ({\sim}Q \to {\sim}P)$ in his system (and this takes work).

- In practice, however, this makes little difference to abbreviated proofs.

  Sider almost immediately helps himself to PL as a derived rule on the grounds that his PL-axioms are complete: they prove every PL-valid formula (given MP).

- Why is it okay to take all MPL-tautologies as axioms?

  This trivialises the propositional part of the proof system. On the other hand MPL-tautologies meet the following desiderata for axioms:

  (i) The axioms are valid (in K).

  (ii) There's an algorithm that delivers a correct 'yes'/'no' answer to the question 'is this formula an axiom?' in a finite amount of time.

Further Exercise 1. Construct abbreviated proofs demonstrating the following:

$$\vdash_{\mathrm{K}} \Box(P \wedge Q) \to \Box P \wedge \Box Q \quad \vdash_{\mathrm{K}} \Box P \vee \Box Q \to \Box(P \vee Q) \quad \vdash_{\mathrm{K}} \Box P \wedge \Box Q \to \Box(P \wedge Q)$$

(See Sider pp. 161–162 for solutions.)

III.9 Proof theory for D, T, B, S4 and S5

We simply add further axioms to those for K. (The definition of S-theorem is as above.)

Axioms and Rules for D, T, B, S4 and S5. D, T, B, S4 and S5 all have all the K axioms and rules, together with the following.

D All instances of $\Box\phi \to \Diamond\phi$ are D-axioms.

T All instances of $\Box\phi \to \phi$ are T-axioms.

B All instances of $\Diamond\Box\phi \to \phi$ and T are B-axioms.

S4 All instances of $\Box\phi \to \Box\,\Box\phi$ and T are S4-axioms.

S5 All instances of $\Diamond\Box\phi \to \Box\phi$ and T are S5-axioms.

Exercise 6. Construct abbreviated proofs to demonstrate the following:

(i) $\vdash_{\mathrm{D}} \Box\phi \to \Diamond\phi$

(ii) $\vdash_{\mathrm{T}} \Box\phi \to \Diamond\phi$

(iii) $\vdash_{\mathrm{K}} {\sim}\Diamond\phi \leftrightarrow \Box{\sim}\phi$

(iv) $\vdash_{\mathrm{K}} \Diamond{\sim}\phi \leftrightarrow {\sim}\Box\phi$

(v) $\vdash_{\mathrm{D}} {\sim}\Box(\phi \wedge {\sim}\phi)$

(vi) $\vdash_{\mathrm{S4}} \Box\phi \to \Box\Diamond\Box\phi$

(vii) $\vdash_{\mathrm{S4}} \Diamond\Diamond\Diamond\phi \to \Diamond\phi$

III.10 Adequacy

Let S be K, D, T, B, S4 or S5. Then S-provability and S-validity coincide.

Soundness theorem If $\vdash_{\mathrm{S}} \phi$, then $\models_{\mathrm{S}} \phi$

Completeness theorem If $\models_{\mathrm{S}} \phi$, then $\vdash_{\mathrm{S}} \phi$

Remark. Soundness is straightforward; Completeness is a bit more involved.