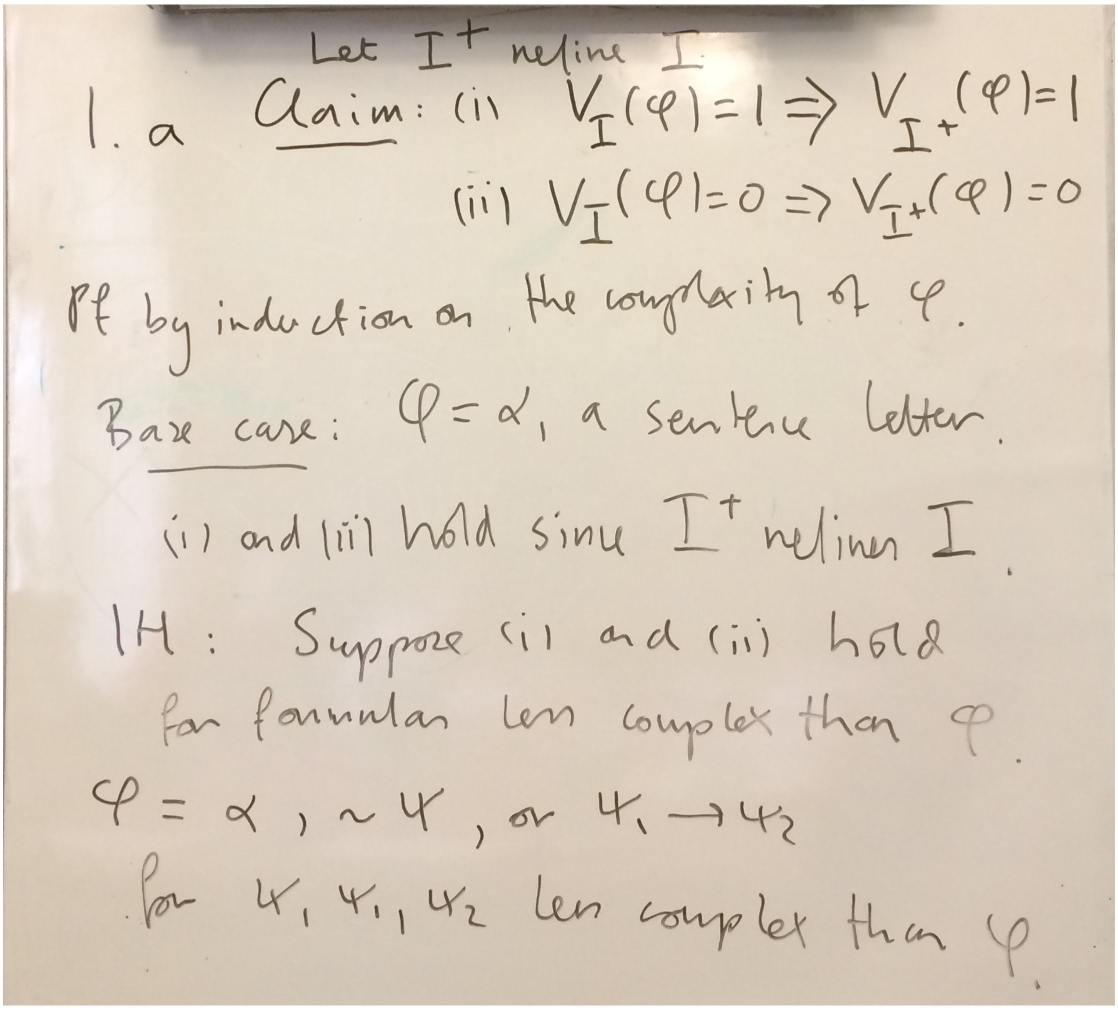
127: TT15 past paper

Board pictures

Revision class TT17

Question 1

(1.a)



(1.a) cont.

Let I^+ refine I

1. a <u>Claim</u>: (i) $KV_I(\varphi)=1 \Rightarrow KV_{I^+}(\varphi)=1$

(ii) $KV_I(\varphi)=0 \Rightarrow KV_{I^+}(\varphi)=0$

• $\varphi = \sim\psi$.

(i) $KV_I(\varphi)=1 \Rightarrow KV_I(\psi)=0$

$$\underset{IH(ii)}{\Rightarrow} KV_{I^+}(\psi)=0$$

$$\Rightarrow KV_{I^+}(\varphi)=1.$$

(ii) $KV_I(\varphi)=0 \Rightarrow KV_I(\psi)=1$

$$\underset{IH(i)}{\Rightarrow} KV_{I^+}(\psi)=1$$

$$\Rightarrow KV_{I^+}(\varphi)=0$$

(1.a) cont.

Let I^+ refine I

1. a <u>Claim</u>: (i) $KV_I(\varphi)=1 \Rightarrow KV_{I^+}(\varphi)=1$

(ii) $KV_I(\varphi)=0 \Rightarrow KV_{I^+}(\varphi)=0$

• $\varphi = \psi_1 \to \psi_2$.

(i) $KV_I(\varphi)=1$

$\Rightarrow KV_I(\psi_1)=0$ or $KV_I(\psi_2)=1$

$\Rightarrow KV_{I^+}(\psi_1)=0$ or $KV_{I^+}(\psi_2)=1$

IH (i),(ii)

$\Rightarrow KV_{I^+}(\varphi)=1$.

(ii) $KV_I(\varphi)=0$

$\Rightarrow KV_I(\psi_1)=1$ and $KV_I(\psi_2)=0$

$\Rightarrow KV_{I^+}(\psi_1)=1$ and $KV_{I^+}(\psi_2)=0$

IH (i),(ii)

$\Rightarrow KV_{I^+}(\varphi)=0$

(1.b.i)

(b.i) Let $D_1 = \{1, \#\}$

<u>Claim</u>: $\models_{PL} \varphi$ iff $\models_{D_1} \varphi$

<u>Pf</u>: ($\Leftarrow$) Suppose $\models_{D_1} \varphi$ but $\not\models_{PL} \varphi$ (for ※)

We have biv. I s.t. $V_I(\varphi) = 0$

By given $KV_I(\varphi) = 0$

$\therefore$ I is a triv. interpretation s.t. $KV_I(\varphi) \notin D_1$.

$\therefore$ ※ $\not\models_{D_1} \varphi$.

(1.b.i) cont.

(b.i) Let $D_1 = \{1, \#\}$

<u>Claim</u>: $\models_{PL} \varphi$ iff $\models_{D_1} \varphi$

<u>Pf:</u> ($\Rightarrow$) Suppose $\models_{PL} \varphi$. But $\not\models_{D_1} \varphi$ (for ※)

$\therefore$ for some triv. I $KV_I(\varphi) \notin D_1$

i.e. $KV_I(\varphi) = 0$

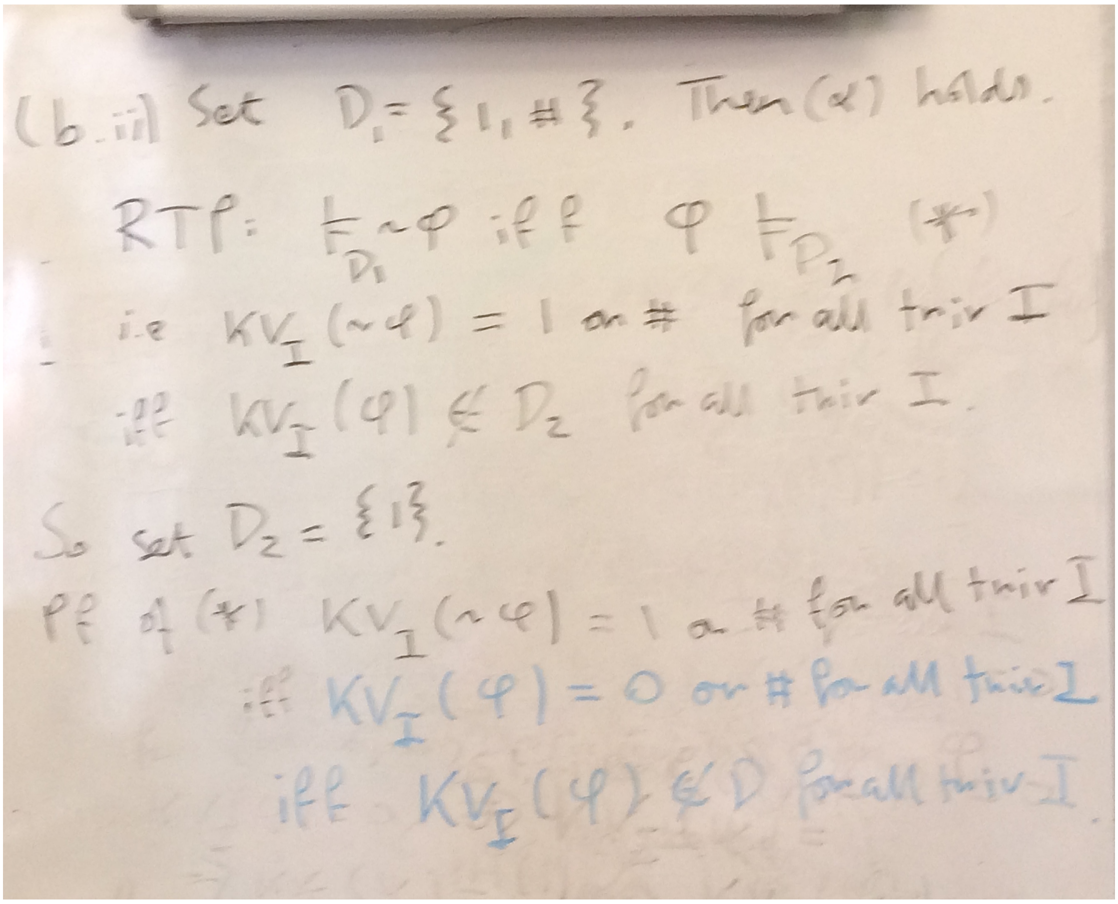
Let biv. I^+ refine I.

Then by (a) $KV_{I^+}(\varphi) = 0$

By given, $V_{I^+}(\varphi) = 0$

※ $\models_{PL} \varphi$.

(1.b.ii)



(b.ii) Set $D_1 = \{1, \#\}$. Then (α) holds.

RTP: $\models_{D_1} \sim\varphi$ iff $\varphi \models_{D_2}$ (*)

i.e. $KV_I(\sim\varphi) = 1$ or $\#$ for all triv I

iff $KV_I(\varphi) \notin D_2$ for all triv I.

So set $D_2 = \{1\}$.

Pf of (*) $KV_I(\sim\varphi) = 1$ or $\#$ for all triv I

iff $KV_I(\varphi) = 0$ or $\#$ for all triv I

iff $KV_I(\varphi) \notin D$ for all triv I.

(1.b.iii)

(b.iii) Suppose (for ※) that D_1 and D_2 satisfy (α), (β), (γ)

$\models_{PL} \sim(p \wedge \sim p)$ (clearly)

By (α) $\models_{D_1} \sim(p \wedge \sim p)$

i.e. $KV_I(\sim(p \wedge \sim p)) \in D_1$ for any triv I

So $\#, 1 \in D_1$

since $KV_I(\sim(p \wedge \sim p)) = 1$ if $I(p) = 1$

and $KV_I(\sim(p \wedge \sim p)) = \#$ if $I(p) = \#$

(1.b.iii) cont.

(b.iii) Suppose (for ※) that D_1 and D_2 satisfy (α), (β), (γ)

Recap: $1, \# \in D_1$

— $\models_{PL} \sim(P \wedge \sim P)$ (clearly

By (α) $\models_{D_1} \sim(P \wedge \sim P)$

By (β) $(P \wedge \sim P) \models_{D_2}$

$\therefore\ KV_I(P \wedge \sim P) \notin D_2$ for any tniv I.

So $\# \notin D_2$

Since $KV_I(P \wedge \sim P) = \#$ if $I(P) = \#$

(1.b.iii) cont.

(b.iii) Suppose (for ※) that D_1 and D_2

satisfy (α), (β), (γ)

Recap: $1, \# \in D_1$, $\# \notin D_2$ ※ $D_1 = D_2$

— $\models_{PL} \sim(P \wedge \sim P)$ (clearly)

By (α) $\models_{D_1} \sim(P \wedge \sim P)$

By (β) $(P \wedge \sim P) \models_{D_2}$

$\therefore$ $KV_I(P \wedge \sim P) \notin D_2$

for any triv I.

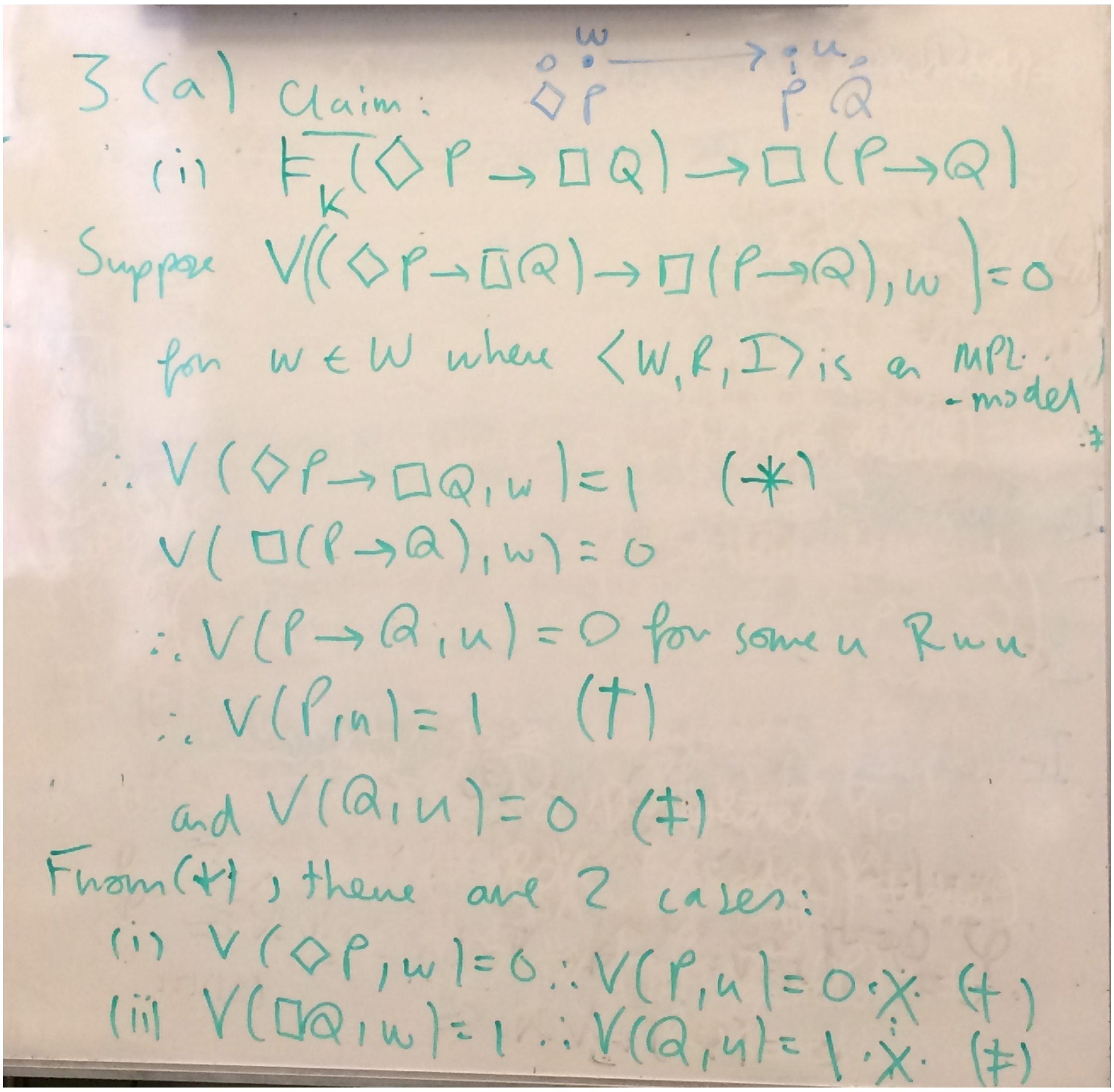
So $\# \notin D_2$

Since $KV_I(P \wedge \sim P) = \#$

if $I(P) = \#$

Question 3

(3.a.i)



3 (a) Claim:

(i) $\vdash_K (\Diamond P \to \Box Q) \to \Box(P \to Q)$

Suppose $V((\Diamond P \to \Box Q) \to \Box(P \to Q), w) = 0$

for $w \in W$ where $\langle W, R, I \rangle$ is an MPL-model

$\therefore V(\Diamond P \to \Box Q, w) = 1$ $(*)$

$V(\Box(P \to Q), w) = 0$

$\therefore V(P \to Q, u) = 0$ for some u Rwu

$\therefore V(P, u) = 1$ $(\dagger)$

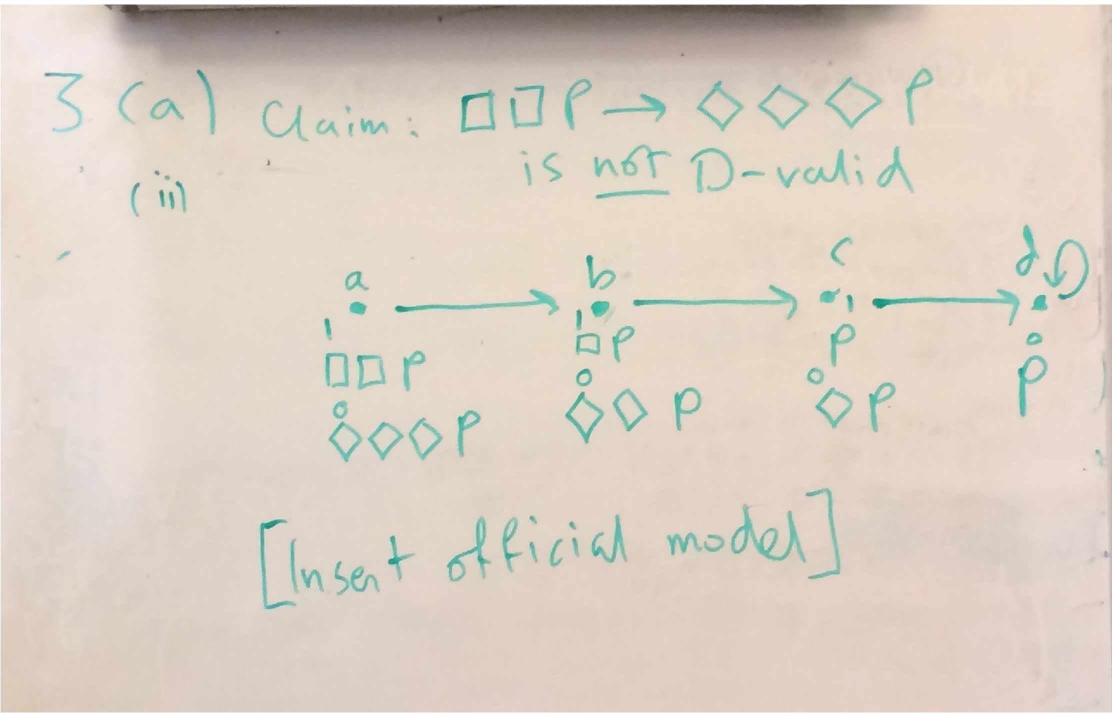
and $V(Q, u) = 0$ $(\ddagger)$

From $(*)$, there are 2 cases:

(i) $V(\Diamond P, w) = 0 \therefore V(P, u) = 0 \therefore \times$ $(\dagger)$

(ii) $V(\Box Q, w) = 1 \therefore V(Q, u) = 1 \therefore \times$ $(\ddagger)$

(3.a.ii)


3 (a) Claim: □□p → ◇◇◇p
is not D-valid
(ii)
a
b
c
d
□□p
◇◇◇p
□p
◇◇p
p
◇p
p
[Insert official model]

(3.a.ii) cont.

3 (a) Claim: $\Box\Box p \to \Diamond\Diamond\Diamond p$ is T valid.

(ii)

Suppose $V(\Box\Box p, w) = 1$ $(*)$

$V(\Diamond\Diamond\Diamond p, w) = 0$ $(\dagger)$

Since Rww, $V(\Box p, w) = 1$ from $(*)$

$\therefore V(p, w) = 1$

$V(\Diamond\Diamond p, w) = 0$ from $(\dagger)$

$\therefore V(\Diamond p, w) = 0$

$\therefore V(p, w) = 0$ $\times$

(3.b.i)

3 (i) By PL, STP

$\therefore \quad \Box Q \to (\Diamond P \to \Diamond(P \wedge Q))$

By PL, STP: $\Box Q \to (\Box \sim(P \wedge Q) \to \Box \sim P)$

<u>Pf</u>: $Q \to (\sim(P \wedge Q) \to \sim P)$ PL

$\Box Q \to \Box(\sim(P \wedge Q) \to \sim P)$ Becker

$\Box(\sim(P \wedge Q) \to \sim P) \to \Box \sim(P \wedge Q) \to \Box \sim P)$

K-axiom

$\Box Q \to (\Box \sim(P \wedge Q) \to \Box \sim P)$

Observe the proof holds also if we uniformly replace P/Q with φ/ψ.

(3.b.ii)

3 (ii)

From our observation in part (i)

$$\vdash_K \Diamond\varphi \wedge \Box\psi \to \Diamond(\varphi \wedge \psi).$$

$$\therefore \vdash_{S4} \Diamond P \wedge \Box\Box Q \to \Diamond(P \wedge \Box Q)$$

$\Box Q \to \Box\Box Q$ S4

$\Diamond P \wedge \Box Q \to (\Diamond P \wedge \Box\Box Q)$ PL

$\Diamond P \wedge \Box Q \to \Diamond(P \wedge \Box Q)$ PL.

(3.b.iii)

3 (iii) Recall $\Diamond\Box\varphi \to \varphi$ $\quad$ B

$\varphi \to \Box\Diamond\varphi$ $\quad$ $B\Diamond$

By Becker, STP: $\Box(p \to \Box p) \to (\Diamond p \to p)$

By PL, STP $(\Diamond p \wedge \Box(p \to \Box p)) \to p$

By (ii) $(\Diamond p \wedge \Box(p \to \Box p)) \to \Diamond(p \wedge (p \to \Box p))$

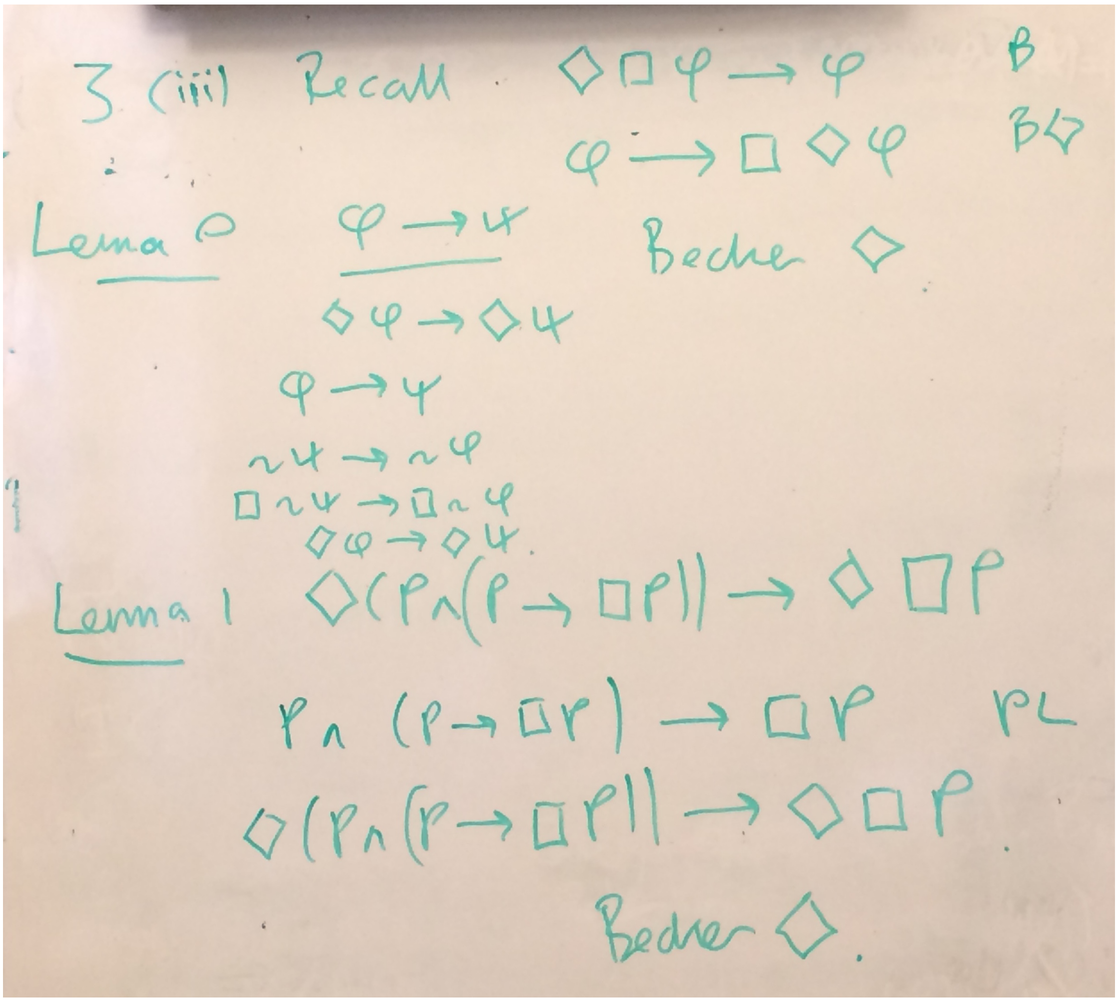
<u>Lemma 1</u> $\Diamond(p \wedge (p \to \Box p)) \to \Diamond\Box p$

$$\Diamond\Box p \to p$$

$$(\Diamond p \wedge \Box(p \to \Box p)) \to p$$

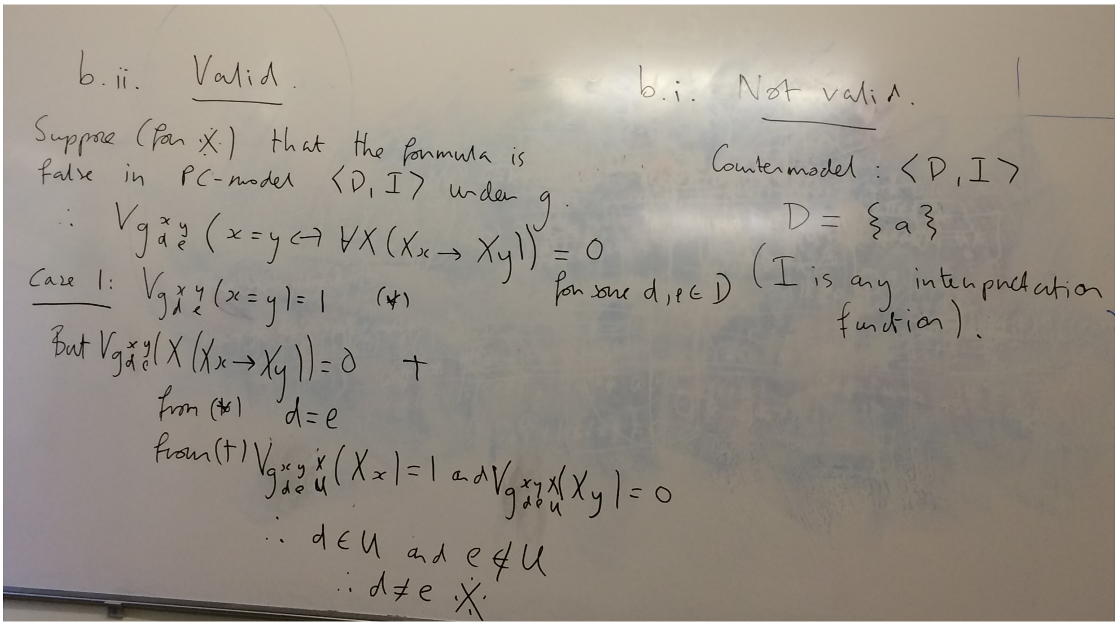
as required.

(3.b.iii) cont.


3 (iii) Recall $\Diamond\Box\varphi \to \varphi$ B
$\varphi \to \Box\Diamond\varphi$ B$\Diamond$
Lemma 0 $\dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$ Becker $\Diamond$
$\varphi \to \psi$
$\sim\psi \to \sim\varphi$
$\Box\sim\psi \to \Box\sim\varphi$
$\Diamond\varphi \to \Diamond\psi$.
Lemma 1 $\Diamond(p \wedge (p \to \Box p)) \to \Diamond\Box p$
$p \wedge (p \to \Box p) \to \Box p$ PL
$\Diamond(p \wedge (p \to \Box p)) \to \Diamond\Box p$.
Becker $\Diamond$.

Question 4

(4.b.i–ii)



(4.b.ii) cont.

(**)

Case 2: $V_{g\,{}^{x\,y}_{d\,e}}(\forall X(Xx \to Xy)) = 1$

but $V_{g\,{}^{x\,y}_{d\,e}}(x = y) = 0$

$\therefore\ d \neq e$

From (**) $V_{g\,{}^{x\,y\,X}_{d\,e\,\{d\}}}(Xx \to Xy) = 1$

Case A: $V_{g\,{}^{x\,y\,X}_{d\,e\,\{d\}}}(Xx) = 0$

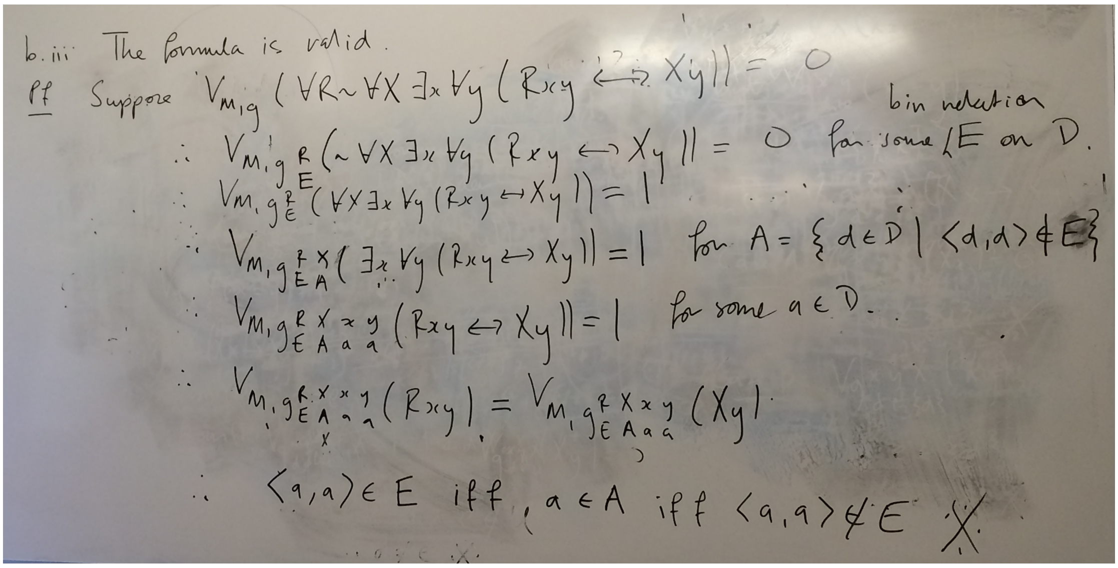
$\therefore\ d \notin \{d\}$ ✗

Case B: $V_{g\,{}^{x\,y\,X}_{d\,e\,\{d\}}}(Xy)$

$\therefore\ e \in \{d\}$

$\therefore\ e = d$ ✗

(4.b.iii)



b.iii The formula is valid.

Pf Suppose $V_{M,g}(\forall R \sim \forall X \exists x \forall y (Rxy \leftrightarrow Xy)) = 0$

∴ $V_{M,g^{R}_{E}}(\sim \forall X \exists x \forall y (Rxy \leftrightarrow Xy)) = 0$ for some bin relation E on D.

∴ $V_{M,g^{R}_{E}}(\forall X \exists x \forall y (Rxy \leftrightarrow Xy)) = 1$

∴ $V_{M,g^{R\,X}_{E\,A}}(\exists x \forall y (Rxy \leftrightarrow Xy)) = 1$ for $A = \{d \in D \mid \langle d,d \rangle \notin E\}$

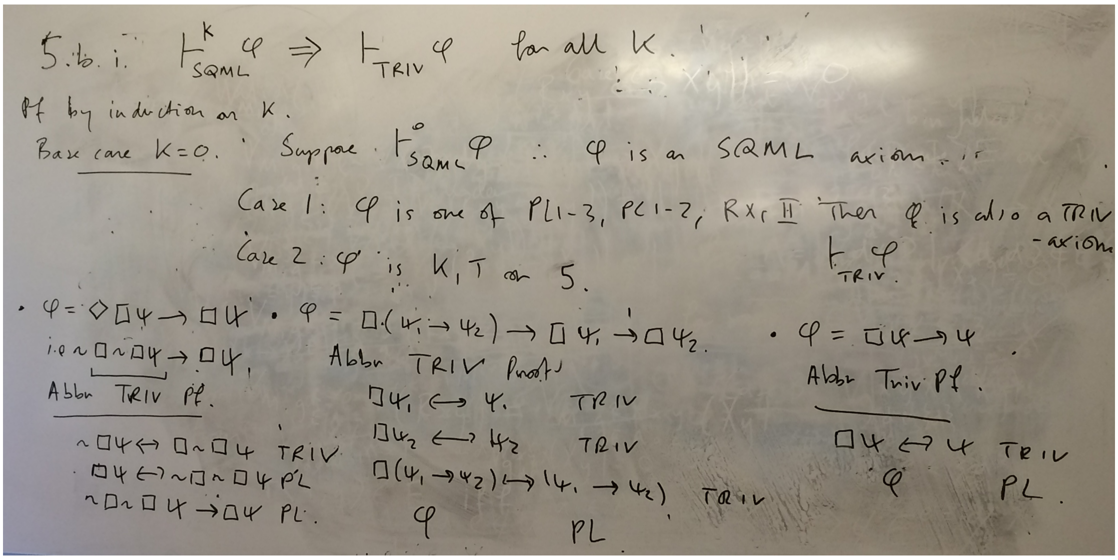
∴ $V_{M,g^{R\,X\,x\,y}_{E\,A\,a\,a}}(Rxy \leftrightarrow Xy) = 1$ for some $a \in D$.

∴ $V_{M,g^{R\,X\,x\,y}_{E\,A\,a\,a}}(Rxy) = V_{M,g^{R\,X\,x\,y}_{E\,A\,a\,a}}(Xy)$

∴ $\langle a,a \rangle \in E$ iff $a \in A$ iff $\langle a,a \rangle \notin E$ $\times$

Question 5

(5.b.i)



5.b.i. $\vdash^k_{SQML} \varphi \Rightarrow \vdash_{TRIV} \varphi$ for all k.

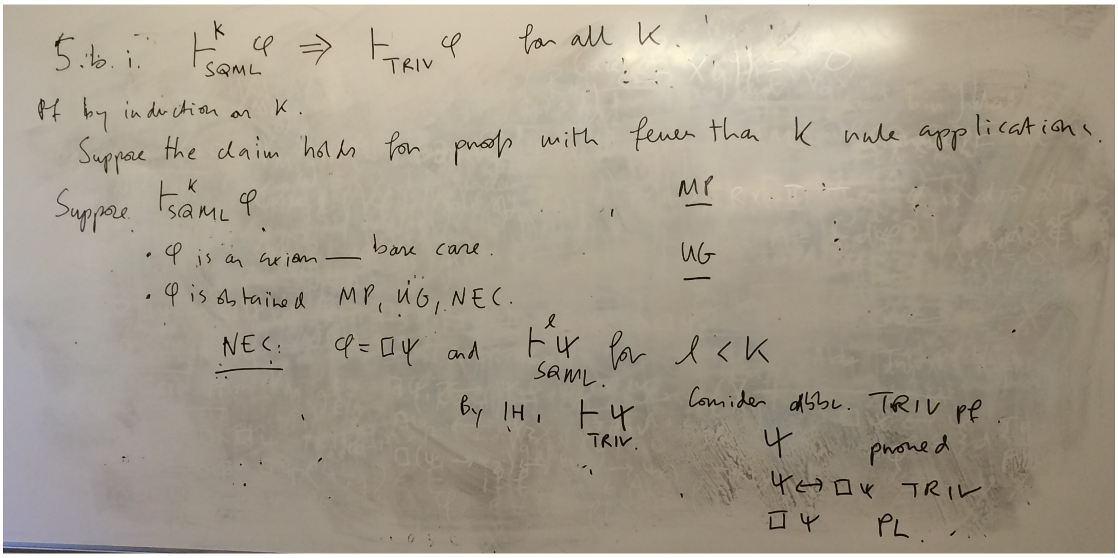
Pf by induction on k.

Base case $k=0$. Suppose $\vdash^0_{SQML} \varphi$ $\therefore$ φ is an SQML axiom.

Case 1: φ is one of PL1-3, PC1-2, RX, II. Then φ is also a TRIV-axiom. $\vdash_{TRIV} \varphi$

Case 2: φ is K, T or 5.

- $\varphi = \Diamond\Box\psi \to \Box\psi$
  i.e. $\sim\Box\sim\Box\psi \to \Box\psi$.
  Abbr TRIV Pf.
  $\sim\Box\psi \leftrightarrow \Box\sim\Box\psi$ TRIV
  $\Box\psi \leftrightarrow \sim\Box\sim\Box\psi$ PL
  $\sim\Box\sim\Box\psi \to \Box\psi$ PL.
- $\varphi = \Box(\psi_1 \to \psi_2) \to \Box\psi_1 \to \Box\psi_2$.
  Abbr TRIV Proof
  $\Box\psi_1 \leftrightarrow \psi_1$ TRIV
  $\Box\psi_2 \leftrightarrow \psi_2$ TRIV
  $\Box(\psi_1 \to \psi_2) \leftrightarrow (\psi_1 \to \psi_2)$ TRIV
  φ PL
- $\varphi = \Box\psi \to \psi$.
  Abbr Triv Pf.
  $\Box\psi \leftrightarrow \psi$ TRIV
  φ PL.



5.b.i. $\vdash^k_{SQML} \varphi \Rightarrow \vdash_{TRIV} \varphi$ for all k.

Pf by induction on k.

Suppose the claim holds for proofs with fewer than k rule applications.

Suppose $\vdash^k_{SQML} \varphi$.

- φ is an axiom — base case.
- φ is obtained MP, UG, NEC.

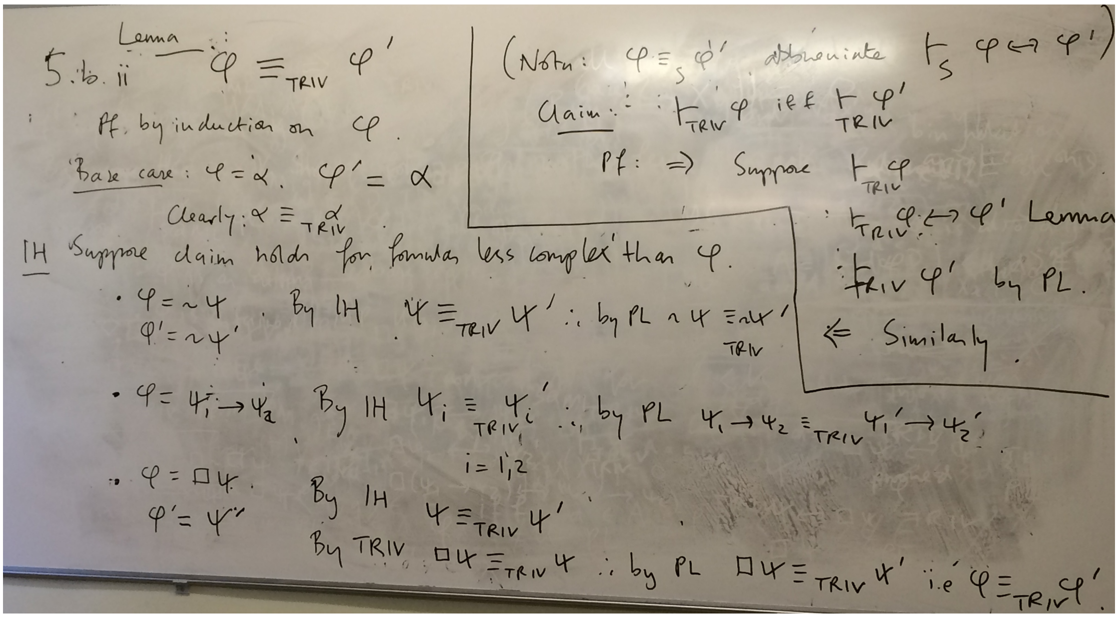
NEC: $\varphi = \Box\psi$ and $\vdash^l_{SQML} \psi$ for $l < k$.

By IH, $\vdash_{TRIV} \psi$. Consider abbr. TRIV pf.
ψ proved
$\psi \leftrightarrow \Box\psi$ TRIV
$\Box\psi$ PL.

MP

UG

(5.b.ii)



5.b.ii Lemma: $\varphi \equiv_{TRIV} \varphi'$
Pf. by induction on φ.
Base case: $\varphi = \alpha$. $\varphi' = \alpha$
Clearly: $\alpha \equiv_{TRIV} \alpha$
IH Suppose claim holds for formulas less complex than φ.
• $\varphi = \sim\psi$, $\varphi' = \sim\psi'$. By IH $\psi \equiv_{TRIV} \psi'$ ∴ by PL $\sim\psi \equiv_{TRIV} \sim\psi'$
• $\varphi = \psi_1 \to \psi_2$ By IH $\psi_i \equiv_{TRIV} \psi_i'$, $i = 1, 2$ ∴ by PL $\psi_1 \to \psi_2 \equiv_{TRIV} \psi_1' \to \psi_2'$
• $\varphi = \Box\psi$, $\varphi' = \psi'$ By IH $\psi \equiv_{TRIV} \psi'$
By TRIV $\Box\psi \equiv_{TRIV} \psi$ ∴ by PL $\Box\psi \equiv_{TRIV} \psi'$ i.e. $\varphi \equiv_{TRIV} \varphi'$.
(Notn: $\varphi \equiv_S \varphi'$ abbreviate $\vdash_S \varphi \leftrightarrow \varphi'$)
Claim: $\vdash_{TRIV} \varphi$ iff $\vdash_{TRIV} \varphi'$
Pf: $\Rightarrow$ Suppose $\vdash_{TRIV} \varphi$
$\vdash_{TRIV} \varphi \leftrightarrow \varphi'$ Lemma
∴ $\vdash_{TRIV} \varphi'$ by PL.
$\Leftarrow$ Similarly.

(5.b.ii) cont.

- $\varphi = \forall\alpha\,\psi$. $\varphi' = \forall\alpha\,\psi'$

By IH $\psi \equiv_{TRIV} \psi'$

$\therefore \forall\alpha\,\psi \equiv_{TRIV} \forall\alpha\,\psi'$ by Lemma 2 and PL.

$\vdash \psi_1 \leftrightarrow \psi_2$
$\vdash \psi_1 \to \psi_2$
$\vdash \forall\alpha\,\psi_1 \to \forall\alpha\,\psi_2$
$\vdash \psi_2 \to \psi_1$
$\vdash \forall\alpha\,\psi_2 \to \forall\alpha\,\psi_1$
$\vdash \forall\alpha\,\psi_1 \leftrightarrow \forall\alpha\,\psi_2$.

Lemma 2 $\dfrac{\chi \to \chi'}{\forall\alpha\,\chi \to \forall\alpha\,\chi'}$ is adissible

Abbr TRIV Pf:

$\chi \to \chi'$
$\forall\alpha\,\chi \to \chi$ PC1
$\forall\alpha\,\chi \to \chi'$ PL
$\forall\alpha(\forall\alpha\,\chi \to \chi')$ UG
$\forall\alpha(\forall\alpha\,\chi \to \chi') \to (\forall\alpha\,\chi \to \forall\alpha\,\chi')$ PC2
$\forall\alpha\,\chi \to \forall\alpha\,\chi'$ PL.

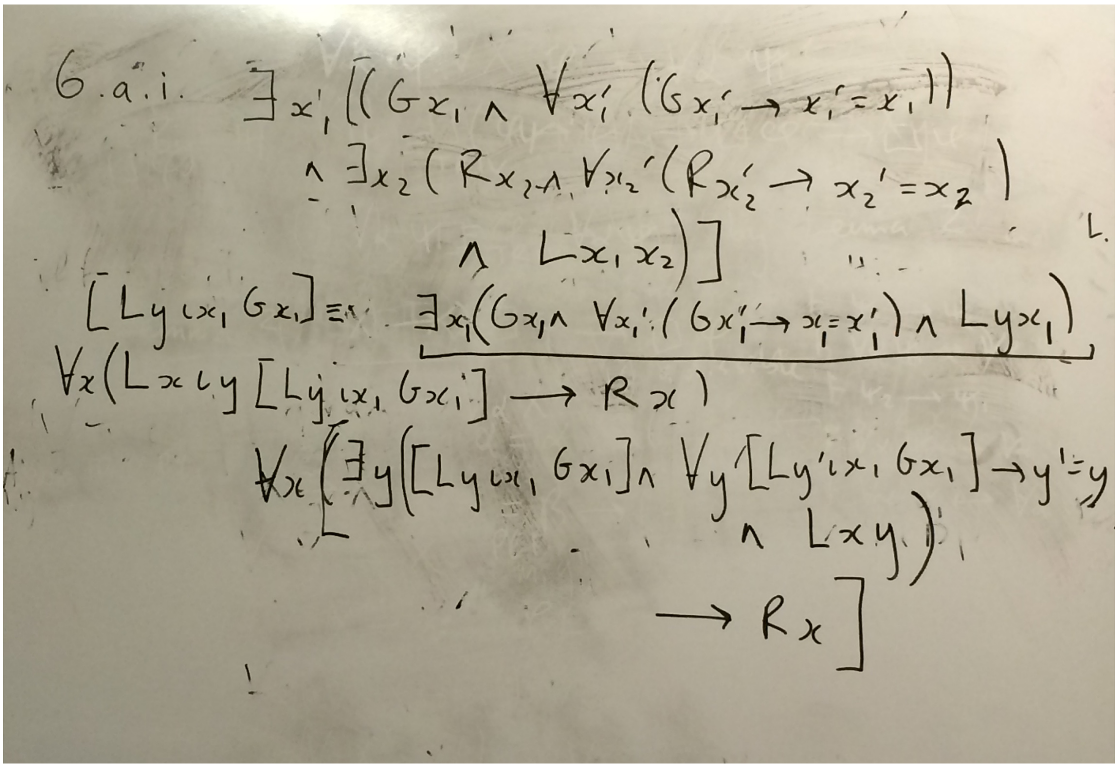
(5.b.iii)

5.b.iii Claim: $\vdash_{TRIV} \varphi$ iff φ is TRIV valid.

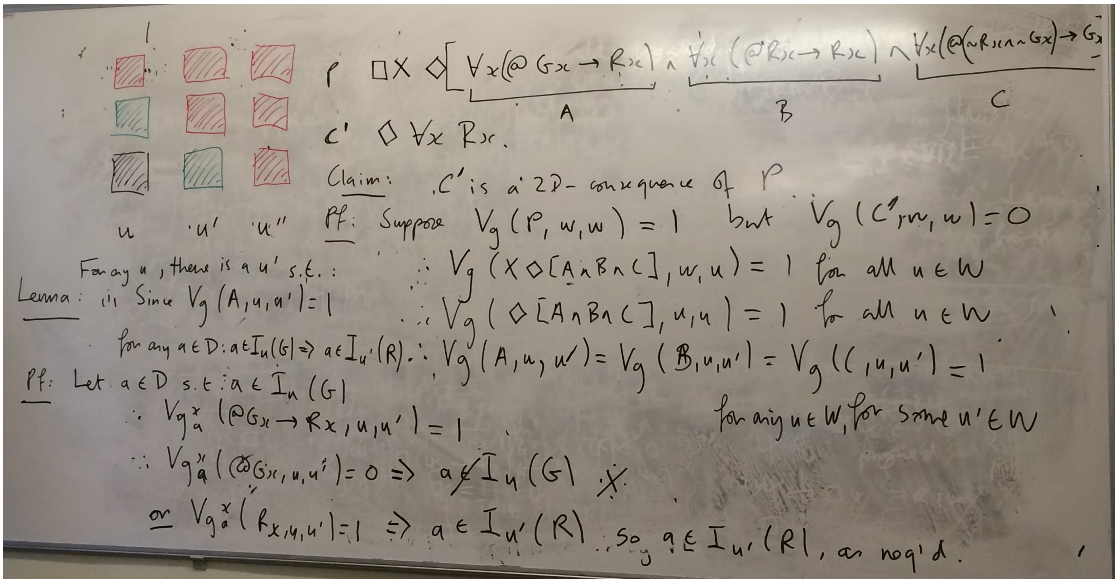
Pf $\vdash_{TRIV} \varphi$ iff (b.ii) $\vdash_{TRIV} \varphi'$
iff $\vDash_{PC} \varphi'$ (Given).
i.e. $V_{M,g}(\varphi') = 1$ for any M and g for M
iff $V^T_{M,g}(\varphi) = 1$ for any M and g for M
i.e. φ is TRIV-valid.

Question 6

(6.a)



(6.b)



(6.b) cont.-

Lemma: By a similar argument.

For any $a \in D$

- $a \in I_u(R) \Rightarrow a \in I_{u'}(R)$

There exists u' s.t. $a \notin I_u(R)$ and $a \notin I_u(G) \Rightarrow I_{u'}(G)$.

Claim 1: for any $a \in D$, $a \in I_{u'}(R)$ or $a \in I_{u'}(G)$

Pf: Let $a \in D$.

Case 1: $a \in I_u(R)$ $\therefore a \in I_{u'}(R)$

Case 2: $a \in I_u(G)$ $\therefore a \in I_{u'}(R)$

Case 3: $a \notin I_u(R)$ and $a \notin I_u(G)$ $\therefore a \in I_{u'}(G)$

There exists u'' s.t.

Claim 2: for any $a \in D$, $a \in I_{u''}(R)$

Case 1: $a \in I_{u'}(R)$ $\therefore a \in I_{u''}(R)$

Case 2: $a \in I_{u'}(G)$ $\therefore a \in I_{u''}(R)$. $\therefore V_g(C, w, w) = 0$