

# 127: TT15 past paper

Board pictures

Revision class TT17

## Question 1

(1.a)

Let  $I^+$  refine  $I$ .

1. a Claim: (i)  $V_I(\varphi) = 1 \Rightarrow V_{I^+}(\varphi) = 1$   
(ii)  $V_I(\varphi) = 0 \Rightarrow V_{I^+}(\varphi) = 0$

Pf by induction on the complexity of  $\varphi$ .

Base case:  $\varphi = \alpha$ , a sentence letter.

(i) and (ii) hold since  $I^+$  refines  $I$ .

IH: Suppose (i) and (ii) hold for formulas less complex than  $\varphi$ .

$\varphi = \alpha$ ,  $\sim \varphi$ , or  $\varphi_1 \rightarrow \varphi_2$   
for  $\varphi_1, \varphi_1, \varphi_2$  less complex than  $\varphi$ .

(1.a) cont.

Let  $I^+$  refine  $I$ .

1. a Claim: (i)  $KV_I(\varphi) = 1 \Rightarrow KV_{I^+}(\varphi) = 1$   
(ii)  $KV_I(\varphi) = 0 \Rightarrow KV_{I^+}(\varphi) = 0$

•  $\varphi \sim \psi$ .

(i)  $KV_I(\varphi) = 1 \Rightarrow KV_I(\psi) = 0$   
 $\Rightarrow KV_{I^+}(\psi) = 0$   
IH(ii)  $\Rightarrow KV_{I^+}(\varphi) = 1$ .

(ii)  $KV_I(\varphi) = 0 \Rightarrow KV_I(\psi) = 1$   
 $\Rightarrow KV_{I^+}(\psi) = 1$   
IH(i)  $\Rightarrow KV_{I^+}(\varphi) = 0$ .

(1.a) cont.

Let  $I^+$  refine  $I$ .

1. a Claim: (i)  $KV_I(\varphi) = 1 \Rightarrow KV_{I^+}(\varphi) = 1$   
(ii)  $KV_I(\varphi) = 0 \Rightarrow KV_{I^+}(\varphi) = 0$

•  $\varphi = \psi_1 \rightarrow \psi_2$ .

(i)  $KV_I(\varphi) = 1$   
 $\Rightarrow KV_I(\psi_1) = 0$  or  $KV_I(\psi_2) = 1$   
 $\Rightarrow KV_{I^+}(\psi_1) = 0$  or  $KV_{I^+}(\psi_2) = 1$   
IH (i), (ii)  $\Rightarrow KV_{I^+}(\varphi) = 1$ .

(ii)  $KV_I(\varphi) = 0 \Rightarrow KV_I(\psi_1) = 1$  and  $KV_I(\psi_2) = 0$   
 $\Rightarrow KV_{I^+}(\psi_1) = 1$  and  $KV_{I^+}(\psi_2) = 0$   
IH (i), (ii)  $\Rightarrow KV_{I^+}(\varphi) = 0$ .

(1.b.i)

(b.i) Let  $D_1 = \{1, \#\}$

Claim:  $\models_{PL} \varphi$  iff  $\models_{D_1} \varphi$

pf: ( $\Leftarrow$ ) Suppose  $\models_{D_1} \varphi$  but  $\not\models_{PL} \varphi$  (con. X.)

We have biv.  $I$  s.t.  $V_I(\varphi) = 0$

By given  $KV_I(\varphi) = 0$

$\therefore I$  is a triv. interpretation

s.t.  $KV_I(\varphi) \notin D_1$

$\therefore \not\models_{D_1} \varphi$

(1.b.i) cont.

(b.i) Let  $D_1 = \{1, \#\}$

Claim:  $F_{PL} \varphi$  iff  $F_{D_1} \varphi$

pf: ( $\Rightarrow$ ) Suppose  $F_{PL} \varphi$ . But  $\nexists \varphi$  (Par.  $\times$ )

$\therefore$  for some triv.  $I$   $KV_I(\varphi) \notin D_1$

i.e.  $KV_I(\varphi) = 0$

Let biv.  $I^+$  refine  $I$ .

Then by (a)  $KV_{I^+}(\varphi) = 0$

By given  $V_{I^+}(\varphi) = 0$

$\times$   
 $F_{PL} \varphi$

(1.b.ii)

(b.ii) Set  $D_1 = \{1, \# \}$ . Then (a) holds.

RTP:  $\vdash_{D_1} \sim \varphi$  iff  $\varphi \vdash_{D_2} (*)$

i.e.  $KV_I(\sim \varphi) = 1$  on  $\#$  for all triv  $I$

iff  $KV_I(\varphi) \notin D_2$  for all triv  $I$ .

So set  $D_2 = \{1\}$ .

Pf of (\*)  $KV_I(\sim \varphi) = 1$  on  $\#$  for all triv  $I$

iff  $KV_I(\varphi) = 0$  on  $\#$  for all triv  $I$

iff  $KV_I(\varphi) \notin D$  for all triv  $I$ .

(1.b.iii)

(b.iii) Suppose (for  $\times$ ) that  $D_1$  and  $D_2$   
satisfy  $(\alpha), (\beta), (\gamma)$

$$F_{PL} \sim (P \wedge \sim P) \text{ (clearly)}$$

$$\text{By } (\alpha) \quad F_{D_1} \sim (P \wedge \sim P)$$

i.e.  $KV_I(\sim(P \wedge \sim P)) \in D_1$  for any  $\text{triv } I$

So  $\#$ ,  $1 \in D_1$

$$\text{Since } KV_I(\sim(P \wedge \sim P)) = 1$$

$$\text{if } I(P) = 1$$

$$\text{and } KV_I(\sim(P \wedge \sim P)) = \#$$

$$\text{if } I(P) = \#$$

(1.b.iii) cont.

(b.iii) Suppose (for  $\times$ ) that  $D_1$  and  $D_2$   
satisfy (A), (B), (C)  
Recap:  $1, \# \in D_1$   
 $\vdash_{PL} \sim (P \wedge \sim P)$  (clearly)  
By (A)  $\vdash_{D_1} \sim (P \wedge \sim P)$   
By (B)  $(P \wedge \sim P) \vdash_{D_2}$   
 $\therefore \text{KV}_I (P \wedge \sim P) \notin D_2$   
for any truth  $I$ .  
So  $\# \notin D_2$   
Since  $\text{KV}_I (P \wedge \sim P) = \#$   
iff  $I(P) = \#$

(1.b.iii) cont.

(b.iii) Suppose (for  $\times$ ) that  $D_1$  and  $D_2$

Satisfy (A), (B), (X)  
Recap:  $\# \in D_1, \# \notin D_2$   $\times$   $D_1 = D_2$   
 $\vdash_{PL} \sim (P \sim P)$  (clearly)

By (A)  $\vdash_{D_1} \sim (P \sim P)$

By (B)  $(P \sim P) \vdash_{D_2}$

$\therefore \text{KV}_I (P \sim P) \notin D_2$

for any triv  $I$ .

So  $\# \notin D_2$

Since  $\text{KV}_I (P \sim P) = \#$

if  $I(P) = \#$

### Question 3

(3.a.i)

3 (a) Claim:  $\diamond P \xrightarrow{w} \begin{matrix} u \\ P \end{matrix} \rightarrow \begin{matrix} u \\ Q \end{matrix}$

(ii)  $\vdash_K (\diamond P \rightarrow \Box Q) \rightarrow \Box (P \rightarrow Q)$

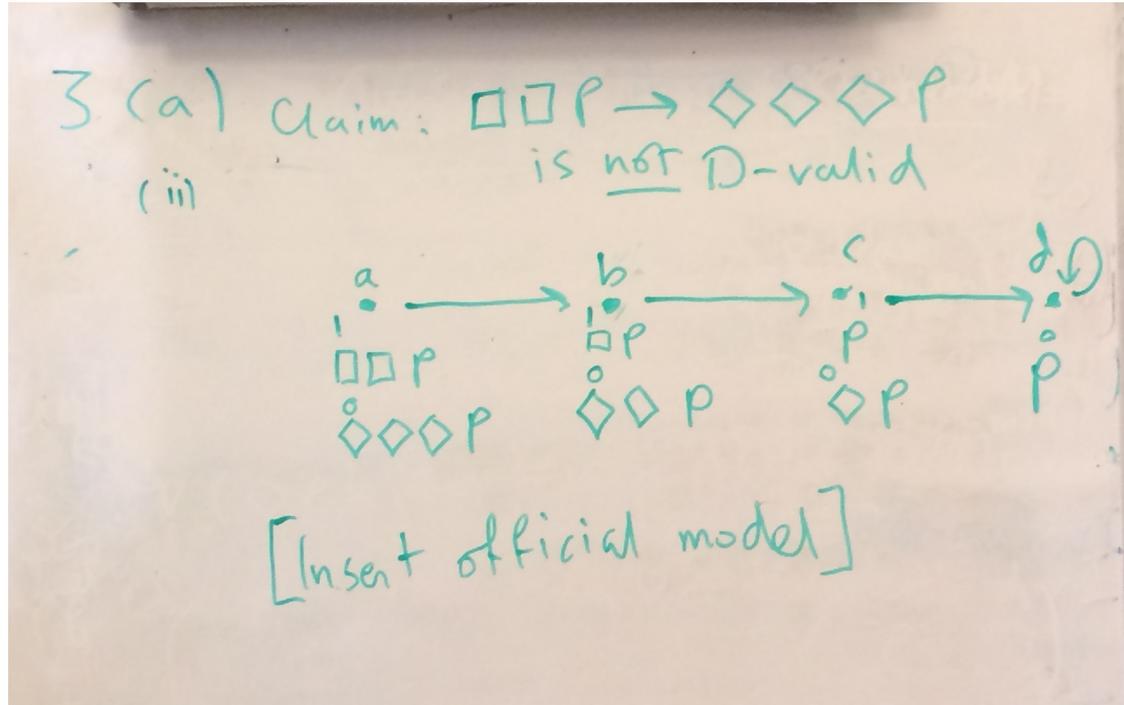
Suppose  $V((\diamond P \rightarrow \Box Q) \rightarrow \Box (P \rightarrow Q), w) = 0$   
 for  $w \in W$  where  $\langle W, R, I \rangle$  is an MPL-model.

$\therefore V(\diamond P \rightarrow \Box Q, w) = 1$  (\*)  
 $V(\Box (P \rightarrow Q), w) = 0$   
 $\therefore V(P \rightarrow Q, u) = 0$  for some  $u R w$   
 $\therefore V(P, u) = 1$  (†)  
 and  $V(Q, u) = 0$  (‡)

From (†), there are 2 cases:

(i)  $V(\diamond P, w) = 0 \therefore V(P, u) = 0$  ✗ (†)  
 (ii)  $V(\Box Q, w) = 1 \therefore V(Q, u) = 1$  ✗ (‡)

(3.a.ii)



(3.a.ii) cont.

3 (a) Claim:  $\Box\Box P \rightarrow \Diamond\Diamond\Diamond P$   
(ii) is T valid.

Suppose  $V(\Box\Box P, w) = 1$  (\*)

$V(\Diamond\Diamond\Diamond P, w) = 0$  (†)

Since  $Rww$ ,  $V(\Box P, w) = 1$  from (\*)

$\therefore V(P, w) = 1$

$V(\Diamond\Diamond P, w) = 0$  from (†)

$\therefore V(\Diamond P, w) = 0$

$\therefore V(P, w) = 0$  ✗

(3.b.i)

3 (i) By PL, STP

$$\dots \Box Q \rightarrow (\Diamond P \rightarrow \Diamond(P \wedge Q))$$

By PL, STP:  $\Box Q \rightarrow (\Box \sim(P \wedge Q) \rightarrow \Box \sim P)$

Pf:  $Q \rightarrow (\sim(P \wedge Q) \rightarrow \sim P)$  PL

$$\Box Q \rightarrow \Box (\sim(P \wedge Q) \rightarrow \sim P) \text{ Becker}$$

$$\Box (\sim(P \wedge Q) \rightarrow \sim P) \rightarrow \Box \sim(P \wedge Q) \rightarrow \Box \sim P$$

K-axiom

$$\Box Q \rightarrow (\Box \sim(P \wedge Q) \rightarrow \Box \sim P)$$

Observe the proof holds also

if we uniformly replace P/Q with  $\psi/\psi$ .

(3.b.ii)

3 (ii)

From our observation in part (i)

$$\vdash_K \Diamond \varphi \wedge \Box \psi \rightarrow \Diamond (\varphi \wedge \psi)$$

$$\therefore \vdash_{S4} \Diamond P \wedge \Box \Box Q \rightarrow \Diamond (P \wedge \Box Q)$$

$$\Box Q \rightarrow \Box \Box Q \quad S4$$

$$\Diamond P \wedge \Box Q \rightarrow (\Diamond P \wedge \Box \Box Q) \quad PL$$

$$\Diamond P \wedge \Box Q \rightarrow \Diamond (P \wedge \Box Q) \quad PL$$

(3.b.iii)

3 (iii) Recall  $\Diamond \Box \varphi \rightarrow \varphi$  B  
 $\varphi \rightarrow \Box \Diamond \varphi$  B4

By Becker, STP:  $\Box(P \rightarrow \Box P) \rightarrow (\Diamond P \rightarrow P)$   
By PL, STP  $(\Diamond P \wedge \Box(P \rightarrow \Box P)) \rightarrow P$   
By (iii)  $(\Diamond P \wedge \Box(P \rightarrow \Box P)) \rightarrow \Diamond(P \wedge (P \rightarrow \Box P))$   
Lemma 1  $\Diamond(P \wedge (P \rightarrow \Box P)) \rightarrow \Diamond \Box P$   
 $\Diamond \Box P \rightarrow P$   
 $(\Diamond P \wedge \Box(P \rightarrow \Box P)) \rightarrow P$   
as required.

(3.b.iii) cont.

$\exists$  (iii) Recall  $\diamond \Box \varphi \rightarrow \varphi$   $\mathcal{B}$   
 $\varphi \rightarrow \Box \diamond \varphi$   $\mathcal{B} \diamond$

Lemma 0  $\frac{\varphi \rightarrow \psi}{\diamond \varphi \rightarrow \diamond \psi}$  Bedner  $\diamond$

$\varphi \rightarrow \psi$   
 $\sim \psi \rightarrow \sim \varphi$   
 $\Box \sim \psi \rightarrow \Box \sim \varphi$   
 $\diamond \varphi \rightarrow \diamond \psi$

Lemma 1  $\diamond (P \wedge (P \rightarrow \Box P)) \rightarrow \diamond \Box P$

$P \wedge (P \rightarrow \Box P) \rightarrow \Box P$  PL  
 $\diamond (P \wedge (P \rightarrow \Box P)) \rightarrow \diamond \Box P$   
 Bedner  $\diamond$

## Question 4

(4.b.i-ii)

b.ii. Valid.

Suppose (for  $x$ ) that the formula is false in PC-model  $\langle D, I \rangle$  under  $g$ .

$\therefore \forall g_{d \neq e} (x=y \leftrightarrow \forall X (Xx \rightarrow Xy)) = 0$

Case 1:  $\forall g_{d \neq e} (x=y) = 1$  (\*)

But  $\forall g_{d \neq e} (X(x \rightarrow Xy)) = 0$  +

from (\*)  $d=e$

from (+)  $\forall g_{d \neq e} (Xx) = 1$  and  $\forall g_{d \neq e} (Xy) = 0$

$\therefore d \in U$  and  $e \notin U$

$\therefore d \neq e$  ✗

b.i. Not valid.

Countermodel:  $\langle D, I \rangle$

$D = \{a\}$

( $I$  is any interpretation function).

(4.b.ii) cont.

(\*\*\*)  
Case 2:  $\forall_{d \neq e} \left( \forall x (x_d \rightarrow x_e) \right) = 1$   
but  $\forall_{d \neq e} (x = y) = 0$

$\therefore d \neq e$

From (\*\*\*)  $\forall_{d \in \{d\}} \forall x (x_d \rightarrow x_e) = 1$

Case A:  $\forall_{d \in \{d\}} \forall x (x_d) = 0$

$\therefore d \notin \{d\} \quad \times$

Case B:  $\forall_{d \in \{d\}} \forall x (x_e)$

$\therefore e \in \{d\}$

$\therefore e = d \quad \times$

(4.b.iii)

b.iii: The formula is valid.  
Pf Suppose  $V_{M, g}(\forall R \sim \forall X \exists x \forall y (Rxy \leftrightarrow Xy)) = 0$  bin relation  
 $\therefore V_{M, g}^R(\sim \forall X \exists x \forall y (Rxy \leftrightarrow Xy)) = 0$  for some  $\langle E \text{ on } D \rangle$ .  
 $\therefore V_{M, g}^E(\forall X \exists x \forall y (Rxy \leftrightarrow Xy)) = 1$   
 $\therefore V_{M, g}^{R \upharpoonright A}(\exists x \forall y (Rxy \leftrightarrow Xy)) = 1$  for  $A = \{d \in D \mid \langle d, d \rangle \notin E\}$   
 $\therefore V_{M, g}^{R \upharpoonright A}(\exists x \forall y (Rxy \leftrightarrow Xy)) = 1$  for some  $a \in D$ .  
 $\therefore V_{M, g}^{R \upharpoonright A}(\exists x \forall y (Rxy \leftrightarrow Xy)) = V_{M, g}^{R \upharpoonright A}(\exists x \forall y (Rxy \leftrightarrow Xy))$   
 $\therefore \langle a, a \rangle \in E \text{ iff } a \in A \text{ iff } \langle a, a \rangle \notin E$  ~~X~~

# Question 5

(5.b.i)

5.b.i.  $\vdash_{SQML}^k \varphi \Rightarrow \vdash_{TRIV} \varphi$  for all  $k$ .

Pf by induction on  $k$ .

Base case  $k=0$ . Suppose  $\vdash_{SQML}^0 \varphi$ .  $\therefore \varphi$  is an SQML axiom.

Case 1:  $\varphi$  is one of PL1-3, PC1-2, RX, H. Then  $\varphi$  is also a TRIV axiom.

Case 2:  $\varphi$  is  $K, T$  or  $S$ .

$\vdash_{TRIV} \varphi$ .

- $\varphi = \Box \Box \psi \rightarrow \Box \psi$ . Abbv TRIV Pf.  $\Box \psi \leftrightarrow \psi$  TRIV.  $\Box \Box \psi \leftrightarrow \Box \psi$  TRIV.  $\Box \Box \psi \rightarrow \Box \psi$  PL.
- $\varphi = \Box (\psi_1 \rightarrow \psi_2) \rightarrow \Box \psi_1 \rightarrow \Box \psi_2$ . Abbv TRIV Pf.  $\Box \psi_1 \leftrightarrow \psi_1$  TRIV.  $\Box \psi_2 \leftrightarrow \psi_2$  TRIV.  $\Box (\psi_1 \rightarrow \psi_2) \leftrightarrow (\psi_1 \rightarrow \psi_2)$  TRIV.  $\Box \psi_1 \rightarrow \Box \psi_2$  PL.
- $\varphi = \Box \psi \rightarrow \psi$ . Abbv Triv Pf.  $\Box \psi \leftrightarrow \psi$  TRIV.  $\Box \psi \rightarrow \psi$  PL.

5.b.i.  $\vdash_{SQML}^k \varphi \Rightarrow \vdash_{TRIV} \varphi$  for all  $k$ .

Pf by induction on  $k$ .

Suppose the claim holds for proofs with fewer than  $k$  rule applications.

Suppose  $\vdash_{SQML}^k \varphi$ .

- $\varphi$  is an axiom — base case.
- $\varphi$  is obtained MP, UG, NEC.

NEC:  $\varphi = \Box \psi$  and  $\vdash_{SQML}^l \psi$  for  $l < k$ .

By IH,  $\vdash_{TRIV} \psi$ .

Consider abbv. TRIV Pf.  $\psi$  proved.  $\psi \leftrightarrow \Box \psi$  TRIV.  $\Box \psi$  PL.

(5.b.ii)

5.b.ii <sup>Lemma</sup>  $\varphi \equiv_{\text{TRIV}} \varphi'$

pf. by induction on  $\varphi$ .

Base case:  $\varphi = \alpha$ ,  $\varphi' = \alpha$   
Clearly:  $\alpha \equiv_{\text{TRIV}} \alpha$

(Note:  $\varphi \equiv_S \varphi'$  abbreviates  $\vdash_S \varphi \leftrightarrow \varphi'$ )

Claim:  $\vdash_{\text{TRIV}} \varphi$  iff  $\vdash_{\text{TRIV}} \varphi'$

Pf:  $\Rightarrow$  Suppose  $\vdash_{\text{TRIV}} \varphi$   
 $\vdash_{\text{TRIV}} \varphi \leftrightarrow \varphi'$  Lemma  
 $\vdash_{\text{TRIV}} \varphi'$  by PL.  
 $\Leftarrow$  Similarly.

IH Suppose claim holds for formulas less complex than  $\varphi$ .

- $\varphi = \neg \psi$ ,  $\varphi' = \neg \psi'$ . By IH  $\psi \equiv_{\text{TRIV}} \psi'$   $\therefore$  by PL  $\neg \psi \equiv_{\text{TRIV}} \neg \psi'$
- $\varphi = \psi_1 \rightarrow \psi_2$ . By IH  $\psi_i \equiv_{\text{TRIV}} \psi'_i$   $\therefore$  by PL  $\psi_1 \rightarrow \psi_2 \equiv_{\text{TRIV}} \psi'_1 \rightarrow \psi'_2$   
 $i = 1, 2$
- $\varphi = \Box \psi$ ,  $\varphi' = \Box \psi'$ . By IH  $\psi \equiv_{\text{TRIV}} \psi'$   
By TRIV:  $\Box \psi \equiv_{\text{TRIV}} \Box \psi'$   $\therefore$  by PL  $\Box \psi \equiv_{\text{TRIV}} \Box \psi'$  i.e.  $\varphi \equiv_{\text{TRIV}} \varphi'$

(5.b.ii) cont.

$\varphi = \forall x \psi$      $\varphi' = \forall x \psi'$   
 By IH  $\psi \equiv_{\text{TRIV}} \psi'$   
 $\forall x \psi \equiv_{\text{TRIV}} \forall x \psi'$  by Lemma 2 and PL.

Lemma 2  $X \rightarrow X'$   
 $\forall x X \rightarrow \forall x X'$  is admissible

Adm TRIV PL:  
 $X \rightarrow X'$   
 $\forall x X \rightarrow X$  PC1  
 $\forall x X \rightarrow X'$  PL  
 $\forall x (\forall x X \rightarrow X')$  VG  
 $\forall x (\forall x X \rightarrow X') \rightarrow (\forall x X \rightarrow \forall x X')$  PC2  
 $\forall x X \rightarrow \forall x X'$  PL

by Lemma 2 and PL:  
 $\vdash \psi_1 \leftrightarrow \psi_2$   
 $\vdash \psi_1 \rightarrow \psi_2$   
 $\vdash \forall x \psi_1 \rightarrow \forall x \psi_2$   
 $\vdash \psi_2 \rightarrow \psi_1$   
 $\vdash \forall x \psi_2 \rightarrow \forall x \psi_1$   
 $\vdash \forall x \psi_1 \leftrightarrow \forall x \psi_2$

(5.b.iii)

5.b.iii Claim -  $\vdash_{\text{TRIV}} \varphi$  iff  $\varphi$  is TRIV valid.

pf  $\vdash_{\text{TRIV}} \varphi$  iff  $\vdash_{\text{TRIV}} \varphi'$   
 b.ii  
 iff  $\vdash_{\text{PC}} \varphi'$  (Given)  
 i.e.  $V_{M,g}(\varphi') = 1$  for any  $M$  and  $g$  for  $M$   
 iff  $V_{M,g}^{\text{TR}}(\varphi) = 1$  for any  $M$  and  $g$  for  $M$   
 i.e.  $\varphi$  is TRIV-valid.

### Question 6

(6.a)

6.a.i. 
$$\exists x_1 \left[ \left( G_{x_1} \wedge \forall x'_1 (G_{x'_1} \rightarrow x'_1 = x_1) \right) \wedge \exists x_2 (R_{x_2} \wedge \forall x'_2 (R_{x'_2} \rightarrow x'_2 = x_2) \wedge L_{x_1, x_2}) \right]$$

$$[L_{y, x_1}, G_{x_1}] \equiv \exists x_1 (G_{x_1} \wedge \forall x'_1 (G_{x'_1} \rightarrow x'_1 = x_1) \wedge L_{y, x_1})$$

$$\forall x (L_{x, y} [L_{y, x_1}, G_{x_1}] \rightarrow R_x)$$

$$\forall x \left( \exists y ([L_{y, x_1}, G_{x_1}] \wedge \forall y ([L_{y, x_1}, G_{x_1}] \rightarrow y' = y) \wedge L_{x, y}) \rightarrow R_x \right)$$

(6.b)

$$P \equiv \exists x \left[ \underbrace{\forall x (G_{x_1} \rightarrow R_{x_1})}_A \wedge \underbrace{\forall x (R_{x_1} \rightarrow R_{x_1})}_B \wedge \underbrace{\forall x (G_{x_1} \rightarrow G_{x_1})}_C \right]$$

$$C' \equiv \forall x R_{x_1}$$

Claim:  $C'$  is a 2D-consequence of  $P$ .

Pf: Suppose  $V_g(P, u, u) = 1$  but  $V_g(C', u, u) = 0$ .

For any  $u$ , there is a  $u'$  s.t.:  $V_g(\exists x [A \wedge B \wedge C], u, u) = 1$  for all  $u \in W$ .

Lemma: (i) Since  $V_g(A, u, u) = 1$ ,  $V_g(\forall x [A \wedge B \wedge C], u, u) = 1$  for all  $u \in W$ .

for any  $a \in D: a \in I_u(G) \Rightarrow a \in I_{u'}(R) \therefore V_g(A, u, u) = V_g(B, u, u) = V_g(C, u, u) = 1$

Pf: Let  $a \in D$  s.t.  $a \in I_u(G)$ .

$\therefore V_g^x(G_{x_1}, u, u) = 1$  for any  $u \in W$ , for some  $u' \in W$

$\therefore V_g^x(G_{x_1}, u, u) = 0 \Rightarrow a \notin I_u(G)$

or  $V_g^x(R_{x_1}, u, u) = 1 \Rightarrow a \in I_{u'}(R)$  so  $a \in I_{u'}(R)$ , as req'd.

(6.b) cont.-

Lemma: By a similar argument.

For any  $a \in D$

$a \in I_u(R) \Rightarrow a \in I_{u'}(R)$

There exists  $u'$  s.t.  $a \notin I_u(R)$  and  $a \notin I_u(G) \Rightarrow I_{u'}(G)$ .

Claim 1: For any  $a \in D$ ,  $a \in I_{u'}(R)$  or  $a \in I_{u'}(G)$

pf: Let  $a \in D$ .

Case 1:  $a \in I_u(R) \therefore a \in I_{u'}(R)$

Case 2:  $a \in I_u(G) \therefore a \in I_{u'}(R)$

Case 3:  $a \notin I_u(R)$  and  $a \notin I_u(G) \therefore a \in I_{u'}(G)$

There exists  $u''$  s.t.

Claim 2: For any  $a \in D$ ,  $a \in I_{u''}(R)$

Case 1:  $a \in I_{u'}(R) \therefore a \in I_{u''}(R)$

Case 2:  $a \in I_{u'}(G) \therefore a \in I_{u''}(R)$   $\times \forall_g (L, w, w) = 0$