

127: TT16 past paper

Board pictures

Revision class TT17

Question 1

(1.a.ii)

TT16

1.a.ii. Suppose (for \ast) $V(\overbrace{\Diamond(\Diamond P \wedge \Diamond \sim P)}^A) \rightarrow \overbrace{\Box(\Diamond P \wedge \Diamond \sim P)}^B, w) = 0$

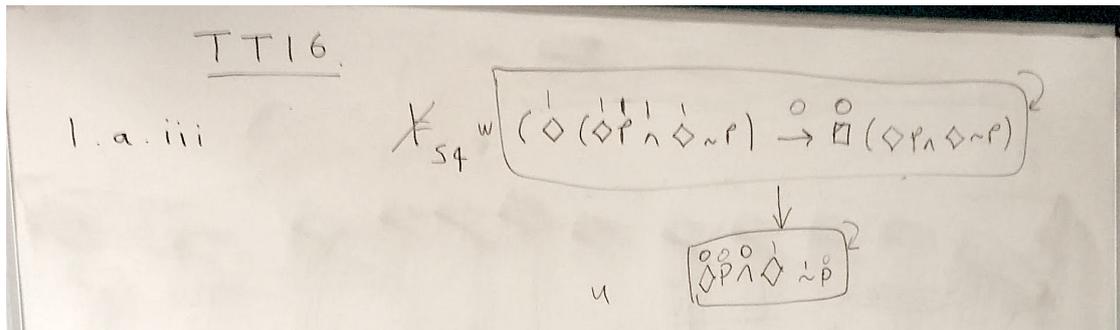
II Since $V(B, w) = 0$ (for some $M = \langle W, R, I \rangle$ SS-model, and $w \in W$)
 for some v s.t. Rwv I Then $V(A, w) = 1$ and $V(B, w) = 0$

$V(\Diamond P \wedge \Diamond \sim P, v) = 0$

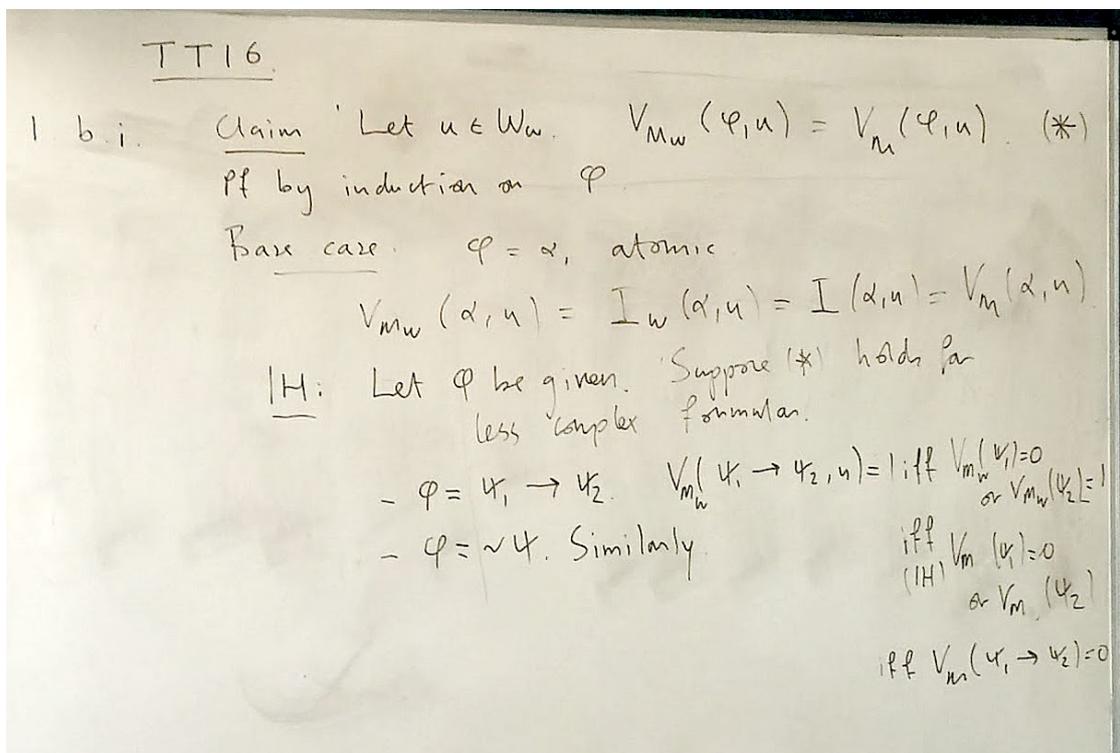
Case 1: $V(\Diamond P, v) = 0$
 Now Rwv , so Rvw (symm)
 Also Rww' , $Rw'w''$
 $\therefore Rvw''$ (trans).
 $\therefore V(P, w'') = 0$
 \ast on \ast

$V(\Diamond P \wedge \Diamond \sim P, w') = 1$ for Rww'
 $\therefore V(\Diamond P, w') = V(\Diamond \sim P, w') = 1$
 $\therefore V(P, w'') = 1$ for $Rw'w''$ (\ast)
 and $V(\sim P, w''') = 1 \therefore V(P, w''') = 0$
 for $Rw'w'''$

(1.a.iii)



(1.b.i)



(1.b.i) cont.

TT16.

1. b. i. Claim 'Let $u \in W_w$. $V_{M_w}(\varphi, u) = V_M(\varphi, u)$. (*)

Pf by induction on φ .

- $\varphi = \Box \psi$. $V_{M_w}(\Box \psi, u) = 1$

To show (**), STP: $\left\{ \begin{array}{l} \{v \in W_w : R_w uv\} \\ \underline{A} = \{v \in W : Ruv\} \\ \underline{B} \end{array} \right.$

(H) iff $V_M(\psi, v) = 1$ for every $v \in W_w$ st. $R_w uv$

(**) iff $V_M(\psi, v) = 1$ for every $v \in W$ st. Ruv

iff $V_M(\Box \psi, u) = 1$

Pf: Let $v \in A$.

Then $v \in W_w = \{v \in W : R_w uv\}$

$\therefore R_w v$.

Also $R_w uv$:

- $\therefore u \in W_w$
- $\therefore R_w u$
- $\therefore R_w v$ (symm)
- $\therefore Ruv$ (tran)
- $\therefore v \in B$.

Let $v \in B$. Then Ruv .

- And $u \in W_w$
- $\therefore R_w u$
- $\therefore R_w v$
- $\therefore v \in W_w$
- $\therefore R_w uv$
- $\therefore v \in A$.

(1.b.ii)

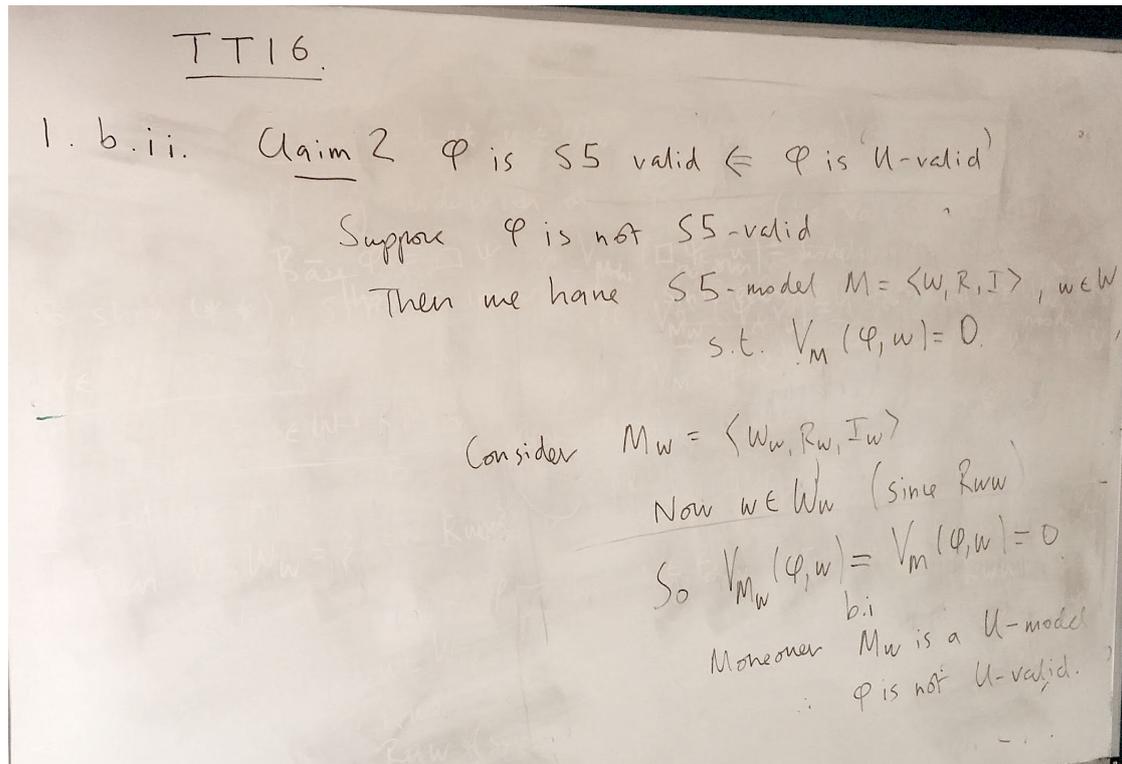
TT16.

1. b. ii. Claim: φ is SS valid $\Rightarrow \varphi$ is U-valid / ie

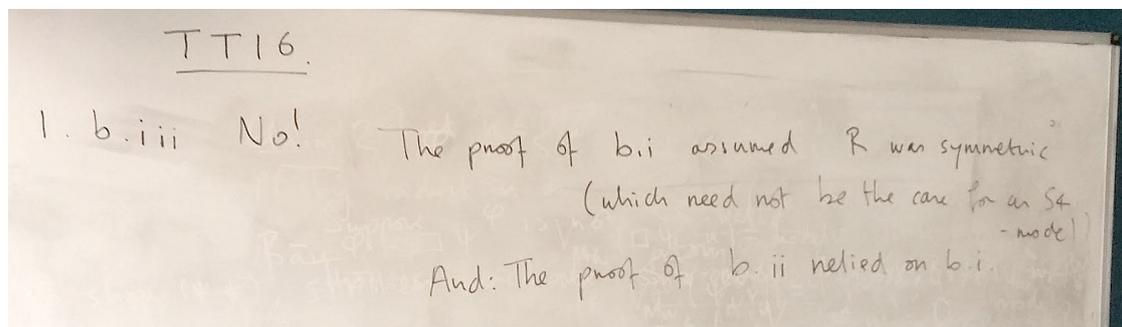
(i.e. valid on all MPL models with universal R)

This is immediate since every U-model is an SS-model

(1.b.ii) cont.



(1.b.iii)



Question 2

(2.a.iii)

TT16.

2.a.iii Claim: If φ is LC-valid, φ is SC-valid.

Note: clearly every SC-model is an LC-model.

SC Base: $w \leq_w u$ for any $u, w \in W$

LC Base: $v \leq_w w$, then $v = w$ for any $v, w \in W$.

When $\langle W, \leq, I \rangle$ is an SC-model, LC Base holds since \leq_w is anti-symmetric.

TT16.

2.a.iii Claim: If φ is LC-valid, φ is SC-valid.

Note: clearly every SC-model is an LC-model.

pf of claim: Suppose φ is not SC-valid.

Then we have $\langle W, \leq, I \rangle$ an SC-model and $w \in W$ s.t. $V(\varphi, w) = 0$.

Lemma: $L.V(\varphi, w) = V(\varphi, w)$.

So, $L.V(\varphi, w) = 0$ and the SC-model is an LC-model.

$\therefore \frac{L.V}{LC} \varphi$.

TT16. For $\langle W, \leq, I \rangle$ an SC-model.

Z. a. iii Lemma: $LV(\varphi, w) = V(\varphi, w)$.

pf by induction. The base case and cases for \neg , \rightarrow , and \Box are routine.

That leaves: $\Box \rightarrow$. RTP: $LV(\varphi \Box \rightarrow \psi, w) = V(\varphi \Box \rightarrow \psi, w)$
(assuming $LV(\varphi, u) = V(\varphi, u)$
 $LV(\psi, u) = V(\psi, u)$.)

Notn: Suppose there is a ' φ -world' u (i.e. $V(\varphi, u) = 1$). Write w_φ for the max. close φ -world to u .

TT16. For $\langle W, \leq, I \rangle$ an SC-model.

Z. a. iii Lemma: $LV(\varphi, w) = V(\varphi, w)$.

pf: Suppose $LV(\varphi \Box \rightarrow \psi, w) = 1$ (*)
 $V(\varphi \Box \rightarrow \psi, w) = 0$ (†) (for \times)

Then there is some φ -world.

Moreover from (*) there is some φ -world u
and $LV(\varphi \rightarrow \psi, v) = 1$ for any $v \leq_w u$ (‡)

From (†) $V(\psi, w_\varphi) = 0$ $\therefore LV(\psi, w_\varphi) = 0$
Moreover $V(\varphi, w_\varphi) = 1$ $\therefore LV(\varphi, w_\varphi) = 1$

$\therefore LV(\varphi \rightarrow \psi, w_\varphi) = 0$

Since w_φ is max close φ -world to w . $w_\varphi \leq_w w$. $\therefore LV(\varphi \rightarrow \psi, w) = 1$.
 \times

TT16.

For $\langle W, \leq, I \rangle$ an SC-model.

2. a. iii

Lemma: $LV(\varphi, w) = V(\varphi, w)$.

Pf: Suppose $LV(\varphi \Box \rightarrow \psi, w) = 0$
 $V(\varphi \Box \rightarrow \psi, w) = 1$ (*)

Then there is a φ -world.

From (*) $V(\psi, w_\varphi) = 1$

Claim: $LV(\varphi, w_\varphi) = 1$ and $LV(\varphi \rightarrow \psi, v) = 1$
for any $v \leq_w w_\varphi$.

Pf: Let $v \leq_w w_\varphi$. If $LV(\varphi, v) = 0$, then $LV(\varphi \rightarrow \psi, v) = 1$
If $LV(\varphi, v) = 1$, then $V(\psi, v) = 1$
Then $w_\varphi \leq_w v$.
Then $v = w_\varphi$.
Then $V(\psi, v) = 1$
Hence $LV(\varphi \rightarrow \psi, v) = 1$.

Question 3

(3.a)

TT16.

3(a) $\Gamma \vDash_{PL} \varphi$ iff $\Gamma \vDash_{SV} \varphi$

\Rightarrow . Suppose $\Gamma \not\vDash_{SV} \varphi$. Then we have
 some triv I s.t. $SV_I(\gamma) = 1 \ \forall \gamma \in \Gamma$
 but $SV_I(\varphi) \neq 1$

So we have biv I^+ , precisifying I
 with $V_{I^+}(\varphi) = 0$.

but $SV_I(\gamma) = 1$ for $\gamma \in \Gamma$
 $\therefore V_{I^+}(\gamma) = 1$
 $\therefore \Gamma \not\vDash_{PL} \varphi$.

(3.b.i)

TT16.

3(b). Note: for a PL-formula φ and for $C \in W$.

Claim: Let φ be a PL-wff. $V_C(\varphi) = V_{M_I}(\varphi, C)$.

Then $SV_I^*(\varphi) = SV_I(\varphi)$ [PF by induction on φ]

$SV_I^*(\varphi) = 1$ iff φ is valid in M_I
 iff $V_{M_I}(\varphi, C) = 1$ for every $C \in W$
 iff $V_C(\varphi) = 1$ for every prec. C of I .
 iff $SV_I(\varphi) = 1$.

(3.b.i) cont.

TT16.

3(b). Note: for a PL-formula φ and for $C \in W$.

Claim: Let φ be a PL-wff. $V_C(\varphi) = V_{M_I}(\varphi, C)$.

Then $SV_I^*(\varphi) = SV_I(\varphi)$. [PF by induction on φ]

$SV_I^*(\varphi) = 0$ iff $\sim\varphi$ is valid in M_I

iff $V_{M_I}(\varphi, C) = 0$ for every $C \in W$

iff $V_C(\varphi) = 0$ for every prec. $C \in I$

iff $SV_I(\varphi) = 0$

(3.b.ii)

TT16.

3(b ii) Claim: if $SV_I^*(\varphi) = 1$, then $SV_I^*(\Box\varphi) = 1$

Pf. Suppose (for \times) $SV_I^*(\varphi) = 1$ (#)

but $SV_I^*(\Box\varphi) \neq 1$.

$\therefore \Box\varphi$ is not valid in M_I .

$\therefore V(\Box\varphi, C) = 0$ for $C \in W$

$\therefore V(\varphi, C') = 0$ for $R \subseteq C'$

$\therefore \varphi$ is not valid in M_I .

$\therefore SV_I^*(\varphi) \neq 1$ ~~(#)~~

(3.b.iii)

TT16.
3(b iii) From b.ii. $P \not\vdash_{S^*} \Box P$

Question: $\vdash_{S^*} P \rightarrow \Box P$? No.

Consider triv I , with $I(P) = \#$
(but $I(\alpha) = 0$ for $\alpha \neq P$)

Consider M_I . s.t. $I_1^+(P) = 1$
Then $W = \{I_1^+, I_0^+\}$ s.t. $I_0^+(P) = 0$

$V_{M_I}(P, I_1^+) = I_1^+(P) = 1$
 $V_{M_I}(\Box P, I_1^+) = 0$ because $V_{M_I}(P, I_0^+) = 0$

TT16.
3(b iii)B From b.ii. $P \not\vdash_{S^*} \Box P$
 $\sim P \not\vdash_{S^*} \Box \sim P$

Question $P \vee \sim P \vdash \Box P \vee \Box \sim P$

Let I be triv with $I(P) = \#$
(and $I(\alpha) = 0$ otherwise)

Consider M_I . s.t. $I_1^+(P) = 1$
Then $W = \{I_1^+, I_0^+\}$ s.t. $I_0^+(P) = 0$

$V_{M_I}(P \vee \sim P, I_1^+) = 1$ (clearly)
 $V_{M_I}(\Box P, I_1^+) = 0$ because $V_{M_I}(P, I_0^+) = 0$
 $V_{M_I}(\Box \sim P, I_1^+) = 0$ because $V_{M_I}(\sim P, I_1^+) = 0$

Question 4

(4.a)

$$4.a.i. \quad \exists x \left[\left(M_{xj} \wedge \forall x' (M_{x'j} \rightarrow x=x') \wedge L_{xj} \right) \wedge \right. \\ \left. \exists y \left(M_{yx} \wedge \forall y' (M_{y'x} \rightarrow y=y') \wedge L_{xy} \right) \right] \\ ii \quad L_{(ix.M_{xj})j} \wedge L_{(ix.M_{xj})(iy.M_{yx}.M_{xj})}$$

(4.b.i)

Notn: $(\exists! x : Fx) \varphi(x) := \exists x (Fx \wedge \forall x' (Fx' \rightarrow x=x') \wedge \varphi(x))$

b.i.

$$P1w (\exists! x : Nx) (\exists! y : @Ny) \diamond Gxy$$

$$P1n \diamond (\exists! x : Nx) (\exists! y : @Ny) Gxy$$

$$P1m (\exists! x : Nx) \diamond (\exists! y : @Ny) Gxy$$

$$P2w (\exists! y : @Ny) \sim \diamond Gay$$

$$P2n \sim \diamond (\exists! y : @Ny) Gay$$

$$Cw (\exists! y : @Ny) \sim y = a$$

$$Cn \sim (\exists! y : @Ny) | y = a.$$

(4.b.ii)

Notn: $(\exists! x: \wp F_x) \varphi(x) := \exists x(\wp F_x \wedge \forall x'(\wp F_{x'} \rightarrow x=x') \wedge \varphi(x))$

b.ii. Claim: $P1_n, P2_w \not\equiv_{2D} C_w$

Countermodel: Let $M = \langle W, D, I \rangle$ s.t.

Then: $V(P1_n, w, w) = 1$ $W = \langle w, w' \rangle$ $D = \{8, 9\}$
 $V(P2_w, w, w) = 1$ $I(a) = 8$
 $V(C_w, w, w) = 0$ for $v=w$ or w' , $I_v(G) = \{\langle 9, 8 \rangle\}$
 $I_w(N) = \{8\}$
 $I_{w'}(N) = \{9\}$

(4.b.iii)

Notn: $(\exists! x: \wp F_x) \varphi(x) := \exists x(\wp F_x \wedge \forall x'(\wp F_{x'} \rightarrow x=x') \wedge \varphi(x))$

Lemma: $V_g((\exists! x: F_x) \varphi(x), v, w) = 1$

iff there is a $d \in D$ s.t. $I_w(F) = \{d\}$
and $V_g^x(\varphi(x), v, w) = 1$

$V_g((\exists! x: \wp F_x) \varphi(x), v, w) = 1$

iff there is a $d \in D$ s.t. $I_v(F) = \{d\}$
and $V_g^x(\varphi(x), v, w) = 1$

(4.b.iii) cont.

Claim 1: $P1_m, P2_n \stackrel{f}{=} C_w$.

Pf: Suppose (for \times) $V_g(P1_m, w, w) = V_g(P2_n, w, w) = 1$
 but $V_g(C_w, w, w) = 0$.

Since $V_g(P1_m, w, w) = 1$ by Lemma,
 there is $d \in D$, s.t. $I_w(N) = \{d\}$.

Since $V_g(P2_n, w, w) = 1$ and $V_g^x(\diamond(\exists! y @ N_y) G_{xy})_{w, w} \stackrel{(*)}{=} 1$
 $V_g(\diamond(\exists! y @ N_y) G_{xy}, w, w) = 0 \neq 1$.

Claim: $d \neq I(a)$. From $(*)$ for some w' $V_g^x(\exists! y @ N_y) G_{xy}, w, w' = 1$
 \therefore for some d' $I_w(N) = \{d'\}$.

From $(\#)$ it's not the case that
 there is some w' and some d' s.t. $I_w(N) = \{d'\}$ and $V_g^x G_{xy}, w, w' = 1$
 and $V_g^y(G_{xy}, w, w') = 1$ $\therefore \langle d, d' \rangle \in I_w(G)$
 i.e. $\langle I(a), d' \rangle \in I_w(G)$. So $d \neq I(a)$.

Claim 1: $P1_m, P2_n \stackrel{f}{=} C_w$.

Pf: Suppose (for \times) $V_g(P1_m, w, w) = V_g(P2_n, w, w) = 1$
 but $V_g(C_w, w, w) = 0$ $(+)$.

Since $V_g(P1_m, w, w) = 1$ by Lemma,
 there is $d \in D$, s.t. $I_w(N) = \{d\}$.

Claim 2 $V_g^y(\neg y = a, w, w) = 1$
 This holds since $d \neq I(a)$
 $V_g^y(y = a, w, w) = 0$
 $\therefore V_g^y(\neg y = a, w, w) = 1$
 \therefore by Lemma $V_g(C_w, w, w) = 1$ $(*)$
 \times $(**)$

Claim: $d \neq I(a)$ \square

Question 5

(5.a.i)

5 a.i. Lemma: $\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$ is admissible.

S. $\varphi \rightarrow \psi$
 $\sim \psi \rightarrow \sim \varphi$ PL
 $\Box \sim \psi \rightarrow \Box \sim \varphi$ Nec, K, MP
 $\sim \Box \sim \varphi \rightarrow \sim \Box \sim \psi$

$\varphi \rightarrow \psi$
 $\Box(\varphi \rightarrow \psi)$ Nec
 $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ K
 $\Box \varphi \rightarrow \Box \psi$ MP

(5.a.ii)

5 a.ii. Claim $\vdash \varphi \rightarrow \Box \Diamond \varphi$

$\Diamond \Box \sim \varphi \rightarrow \Box \sim \varphi$ S
 $\Box \sim \varphi \rightarrow \sim \varphi$ T
 $\Diamond \Box \sim \varphi \rightarrow \sim \varphi$ PL
 $\sim \sim \varphi \rightarrow \sim \sim \Box \sim \Diamond \sim \varphi$ PL
 $\varphi \rightarrow \Box \Diamond \varphi$ PL

(5.a.iii)

5 a.ii. Claim $\vdash \Box \exists \alpha \varphi \rightarrow \exists \alpha \Box \varphi$
 i.e. $\sim \Box \sim \forall \alpha \varphi \rightarrow \sim \forall \alpha \sim \Box \sim \varphi$

By PL, STP $\vdash \forall \alpha \sim \Box \sim \varphi \rightarrow \Box \sim \forall \alpha \sim \varphi$

$\forall \alpha \sim \Box \sim \varphi \rightarrow \sim \Box \sim \varphi$ PC1
 $\forall \alpha \sim \Box \sim \varphi \rightarrow \Box \sim \varphi$..
 $\Box \forall \alpha \sim \Box \sim \varphi \rightarrow \Box \Box \sim \varphi$ (a.i)
 $\Box \Box \sim \varphi \rightarrow \sim \varphi$ (a.ii)
 $\Box \forall \alpha \sim \Box \sim \varphi \rightarrow \sim \varphi$ PL
 $\Box \forall \alpha \sim \Box \sim \varphi \rightarrow \forall \alpha \sim \varphi$ UG, P2, MP
 $\Box \forall \alpha \sim \Box \sim \varphi \rightarrow \sim \forall \alpha \varphi$ PL
 $\Box \Box \forall \alpha \sim \Box \sim \varphi \rightarrow \Box \sim \forall \alpha \varphi$ (a.i)
 $\forall \alpha \sim \Box \sim \varphi \rightarrow \Box \Box \forall \alpha \sim \Box \sim \varphi$ (a.ii)
 $\forall \alpha \sim \Box \sim \varphi \rightarrow \Box \sim \forall \alpha \varphi$ PL

(5.b.i)

5.b.i Claim $\vdash_{\forall \alpha \text{ in } \mathcal{L}} \text{FPC}$ (for \mathcal{X})

Suppose $\forall \beta (\forall \alpha \varphi \rightarrow \varphi(\beta/\alpha), w) = 0$

Then: $\forall \beta \forall \alpha (\forall \alpha \varphi \rightarrow \varphi(\beta/\alpha), w) = 0 \quad \alpha \in D_w$

$\therefore \forall \beta \forall \alpha (\forall \alpha \varphi, w) = 1 \therefore \forall \beta \forall \alpha \forall \alpha (\varphi, w) = 1$ (*)

and $\forall \beta \forall \alpha (\varphi(\beta/\alpha), w) = 0$

$\therefore \forall \beta \forall \alpha \forall \alpha (\varphi(\beta/\alpha), w) = 0$ (α does not occur free).

Since $[\alpha] \forall \beta \forall \alpha \forall \alpha (\varphi, w) = [\beta] \forall \beta \forall \alpha \forall \alpha (\varphi, w)$ we have $\forall \beta \forall \alpha \forall \alpha (\varphi, w) = 0$ ~~✗~~ (*)

(5.b.ii)

b.ii. The class of $\forall\exists$ DAML models whose accessibility rel R is an eq. rel on W .
 $K, T, S5$ are valid in all such models.
The other axioms are valid in all $\forall\exists$ DAML-models.

(5.b.iii)

5 b.iii. Note: the rules preserve validity in a model.
So: $\vdash_{\text{FDAML}} \varphi \Rightarrow \varphi$ is valid in all models in the class from b.ii.
Countermodel $\not\models \Diamond \exists x Fx \rightarrow \exists x \Diamond Fx$
 $w \curvearrowright \boxed{} \quad D_w = \emptyset$
 $v \curvearrowright \boxed{} \quad D_w = \{0\}$
 $I_w(F) = \{0\}$
So by Soundness $\not\models_{\text{FDAML}} \Diamond \exists x Fx \rightarrow \exists x \Diamond Fx$