

Write $I^+ > I$ if I^+ preserves the classical
TVS assigned by I .

1. a. i. Proof by induction that φ multi S under KV .
i.e. if $I^+ > I$, then $KV_{I^+} > KV_I$.

Base case: $\varphi = \alpha$, a S.L.

$$KV_{I^+}(\varphi) = I^+(\alpha) \quad \text{1-2 divide.}$$

$$KV_I(\varphi) = I(\alpha).$$

Consider complex φ .

IH: Suppose S holds for all ψ less complex than φ .

$\varphi = \sim \psi$. Suppose $I^+ > I$.

$$\text{Suppose } KV_I(\varphi) = 1$$

$$\therefore KV_I(\psi) = 0$$

$$\therefore KV_{I^+}(\psi) = 0$$

$$\text{IH } KV_{I^+}(\psi) = 1$$

$$\therefore KV_{I^+}(\varphi) = 0$$

Similarly, if $KV_I(\varphi) = 0$, then
 $KV_{I^+}(\varphi) = 0$.

$\varphi = \psi_1 \rightarrow \psi_2$. Suppose $I^+ > I$.

$$\text{Suppose } KV_I(\varphi) = 1.$$

$$\therefore KV_I(\psi_1) = 0 \text{ or } KV_I(\psi_2) = 1$$

$$\therefore KV_{I^+}(\psi_1) = 0 \text{ or } KV_{I^+}(\psi_2) = 1$$

$$\text{IH } KV_{I^+}(\varphi) = 1.$$

Similarly, if $KV_I(\varphi) = 0$,
then $KV_{I^+} = 0$.

a.ii. Consider $\varphi = P \rightarrow Q$

Let $I(P) = I(Q) = \#$ then $\vDash_{V_I}(\varphi) = 1$.

But there is $I^+ > I$ s.t. $I(P) = 1$ and $I(Q) = 0$

$$\therefore \vDash_{V_{I^+}}(\varphi) = 0.$$

So S fails for $\vDash V$.

b. Claim: $\sim$ has the following truth table under XV :

φ	$\sim \varphi$
1	0
0	1
#	#

Suppose $\varphi = P$. Let I^+ be biv.

$$XV_{I^+}(\sim P) \stackrel{F}{=} V_{I^+}(\sim P) = \begin{cases} 1 & \text{iff } XV_{I^+}(P) \stackrel{F}{=} V_{I^+}(P) = 0 \\ 0 & \text{iff } XV_{I^+}(P) \stackrel{F}{=} V_{I^+}(P) = 1 \end{cases}$$

By truth functionality $XV_I(\sim \varphi) = N(XV_I(\varphi))$.
 Suppose $I^+(P) = 1$. $N: \{0, 1, \#\} \rightarrow \{0, 1, \#\}$.

$$\begin{aligned} N(1) &= N(XV_{I^+}(P)) \\ &= XV_{I^+}(\sim P) = 0. \end{aligned} \quad \left. \begin{aligned} N(0) &= N(XV_{I^+}(P)) = XV_{I^+}(\sim P) = 1. \end{aligned} \right\} \text{as above.}$$

Suppose $I^+(P) = 0$. $N(0) = N(XV_{I^+}(P)) = XV_{I^+}(\sim P) = 1.$

Suppose $XV_I(\varphi) = \#$ $\varphi = P$.

Suppose $XV_I(\sim\varphi) = 0$ (for X).

Let $I^+ > I$ s.t. $I^+(P) = 0$ be bivalent.

$XV_{I^+}(\sim P) \underset{F}{=} V_{I^+}(\sim P) = 1$ ~~X~~ S So $XV_I(\sim\varphi) \neq 0$

Suppose $XV_I(\sim\varphi) = 1$ (for X)
 Let $I^+ > I$ s.t. $I^+(P) = 1$ be bivalent.
 $XV_{I^+}(\sim P) \underset{F}{=} V_{I^+}(\sim P) = 0$ ~~X~~ S] Similarly,
 $XV_I(\sim\varphi) \neq 1$

By truth functionality: $N(\#) = \#$
 we obtain the bottom row of
 the truth table.

b.i. By truth-functionality and condition F

$$\text{if } X_{V_I}(\varphi) = X_{V_{I^+}}(P) = 0 \text{ or } 1$$

$$\text{and } X_{V_I}(\psi) = X_{V_{I^+}}(Q) = 0 \text{ or } 1.$$

$$\text{then } X_{V_I}(\varphi \wedge \psi) = X_{V_{I^+}}(P \wedge Q) \stackrel{F}{=} V_{I^+}(P \wedge Q)$$

So the truth table for $\wedge$ (under X_V) is as follows

	$\wedge$	1	0	#
P	1	1	0	e
	0	0	0	d
	#	a	b	c

Claim 1: $a, b, c \neq 1$.

pf. Suppose $I(P) = \#$

$$\text{Suppose } X_{V_I}(P \wedge Q) = 1 \text{ (for } \times \text{)}$$

Let $I^+ \succ I$ s.t. $I(P) = 0$.

$$X_{V_{I^+}}(P \wedge Q) \stackrel{F}{=} V_{I^+}(P \wedge Q) = 0$$

~~$\times$~~ S.
□ Claim 1.

Claim 2: $c, d, e \neq 1$.
pf. Similarly.

Claim 3: $a, b, c \neq 0$.

Suppose $I(P) \neq 0$ and $I(Q) \neq 0$
[and $I(P) = \#$ or $I(Q) = \#$]

$$\text{Suppose } X_{V_I}(P \wedge Q) = 0$$

Let biv. $I^+ \succ I$. $I^+(P) = 1$ and $I^+(Q) = 1$

$$\text{Then } X_{V_{I^+}}(P \wedge Q) = V_{I^+}(P \wedge Q) = 1$$

~~$\times$~~ S.

not strictly
meaning.

This leaves the 4 truth-tables we've given.

Let φ be a MPL-wff.

2. (a) Claim: $md(\varphi) = 0$ iff φ is a PL-wff.

Pf by induction on φ .

Base case: $\varphi = \alpha$, a S.L.

$md(\varphi) = 0$ and φ is PL-wff.

IH: Suppose claim holds for formulas less complex than φ .

$\varphi = \sim \psi$.

$md(\varphi) = 0$ iff $md(\psi) = 0$ iff ψ is a PL-wff
IH iff $\sim \psi$ is a PL-wff.

$\varphi = \psi \rightarrow \chi$

$md(\varphi) = 0$ iff $md(\psi) = md(\chi) = 0$

iff ψ, χ are PL-wffs

iff φ is a PL-wff

$\varphi = \Box \psi$.

$md(\varphi) \neq 0$ and φ is not a PL-wff.

Let φ be a PL-wff.

a.ii. Claim, $V_m(\varphi, w) = V_{Iw}(\varphi)$.

pf by induction:

• $\varphi = \alpha$. $V_m(\varphi, w) = I(\alpha, w) = I^w(\alpha) = V_{Iw}(\varphi)$.

IH: Suppose claim holds for wffs less complex than φ .

• $\varphi = \psi \rightarrow \chi$

$V_m(\varphi, w) = 0$

iff $V_m(\psi, w) = 1$ AND $V(\chi, w) = 0$

iff $V_{Iw}(\psi) = 1$ AND $V_{Iw}(\chi) = 0$

iff $V_{Iw}(\varphi) = 0$.

• $\varphi = \sim \psi$. Similarly.

b.i. Suppose φ is valid in T .

Let M be an SS -model.

M is also a T -model

$\therefore \varphi$ is valid in M .

So, since M is an arbitrary SS model.

φ is valid in SS .

Suppose Claim holds for formulas less complex than φ .

b.ii. Suppose φ is not valid in T . s.t. $md(\varphi) \leq 1$.

∴ there is a T -model $m = \langle W, R, I \rangle$ and $w \in W$
s.t. $V_m(\varphi, w) = 0$.

Define $m' = \langle W', R', I' \rangle$
 $W' = \{u : Rww\}$ (note $w \in W'$ since R is reflexive on w)
 $R' = W' \times W'$

$I'(x, u) = I(x, u)$ for $u \in W'$

Claim: $V_{m'}(\varphi, w) = V_m(\varphi, w)$
Suppose $md(\varphi) \leq 1$

pf by induction on φ .

• $\varphi = \alpha$, a S.L. Base case
holds by df of I' .

$\varphi = \varphi_1 \rightarrow \varphi_2$

$V_{m'}(\varphi, w) = 0$

iff $V_{m'}(\varphi_1, w) = 1$ and $V_{m'}(\varphi_2, w) = 0$

iff $V_m(\varphi_1, w) = 1$ and $V_m(\varphi_2, w) = 0$

iff

$V_m(\varphi_1, w) = 0$

$\varphi = \sim x$, similarly

$\varphi = \Box x$. $V_{m'}(\Box x, w) = 1$ note $md(x) = 0$
iff for all $u \in W'$ $V_{m'}(x, u) = 1$
iff for all $u \in W'$ $V_{I'}^u(x) = 1$
a.ii $V_{I'}^u(x) = 1$
iff $V_m(x, u) = 1$
a.ii $V_m(x, u) = 1$

(*) iff

iff

a.ii

(**) iff for all u s.t. Rwu $V_m(x, u) = 1$
iff $V_m(\Box x, w) = 1$.

(*) since $(I')^u(\alpha) = I'(\alpha, u) = I(\alpha, u) = I^u(\alpha)$
by df of W' .

From claim: $V_{m'}(\varphi, w) = 0$

∴ φ is not SS-valid.

Let $I := (x=1, x=1)(x=x)$
 $\exists a. \forall m. V_{m, \gamma}(I) = 1 \text{ iff } D \text{ is infinite for } M = (D, I)$
 $(x=1)^{m, \gamma} = D$

Suppose for X , I is a p.c. wff.

Consider $\Delta = \{\sim I\} \cup \{E_n : n > 0\}$

$$E_n := \exists x_1 \dots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i = x_j$$

Claim 1. Δ is finitely satisfiable.

Let Δ_0 be a finite subset of Δ

Then $\Delta_0 \subseteq \{\sim I, E_1, \dots, E_N\}$ for $N \in \mathbb{N}$.

Δ_0 has a model with domain $\{1, \dots, N\}$

Claim 2: Δ is not satisfiable.
 $= (D, I)$

Suppose for X , M satisfies Δ

then M satisfies $\sim I$

$|D| = m$ for $m \in \mathbb{N}$

But then E_{m+1} is false in M

~~M satisfies Δ .~~

From Claim 1 and 2, compactness fails X .

b.i. Suppose the sentence is false in $M = (D, I)$.

Then $(\exists x: P_x \wedge Q_x) \wedge P_x$ is true in M (*)

and $(\exists x: Q_x \wedge P_x) \wedge Q_x$ is false in M (**)

$$P_x^{M, g, x} = \bar{P} := I(P)$$

$$(P_x \wedge Q_x)^{M, g, x} = \bar{P} \cap \bar{Q}$$

$$(\sim P_x)^{M, g, x} = D \setminus \bar{P}$$

Similarly for $Q_x, Q_x \wedge P_x, \sim Q_x$

From (*), $\bar{P} \cap \bar{Q}$ is infinite.

and $\emptyset$ is finite.

From (**), $\bar{P} \cap \bar{Q}$ is finite

or $\emptyset$ is infinite.

Either way, $\bar{P} \cap \bar{Q}$.

$\therefore$ The sentence is valid.

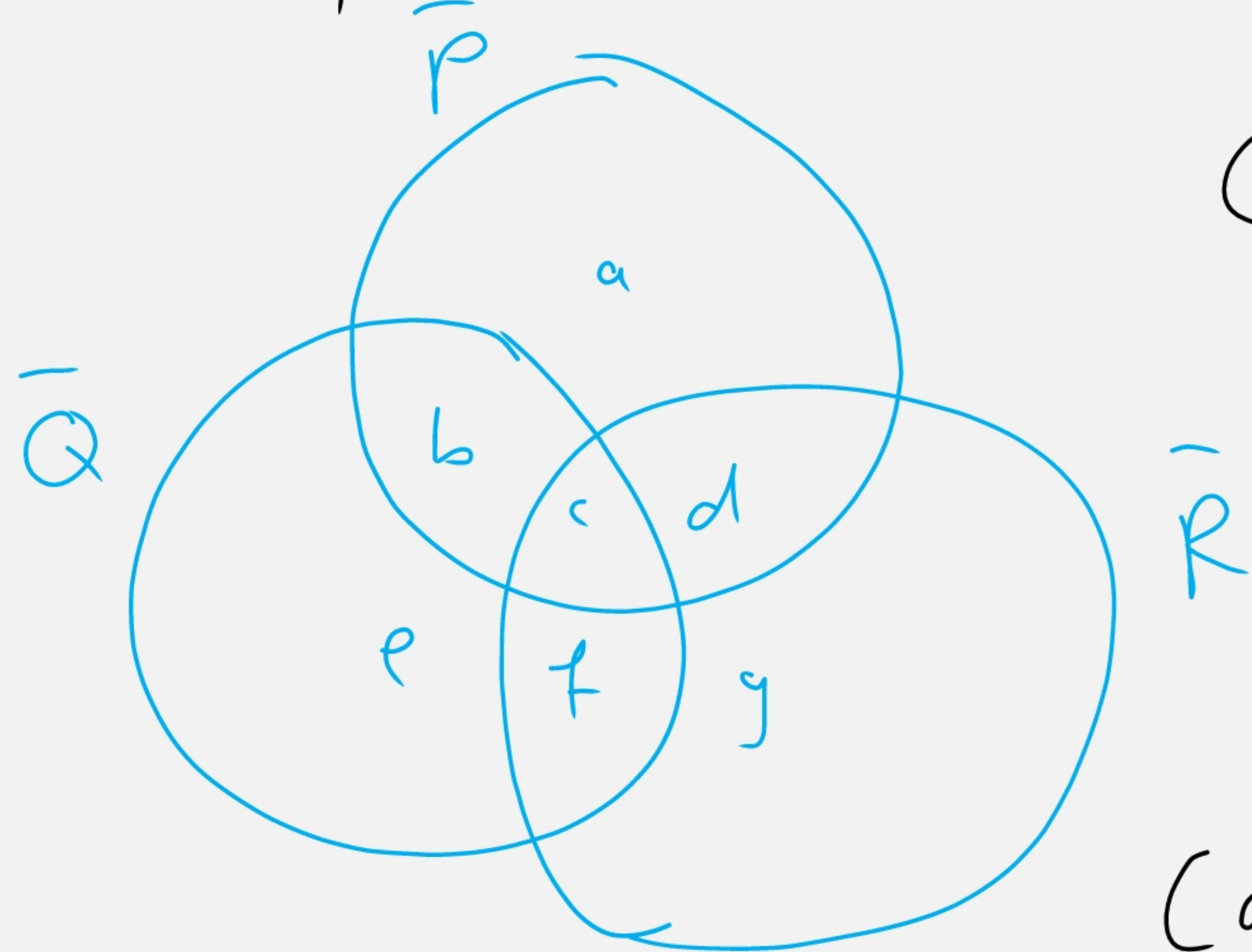
b.ii. Suppose the sentence is false in $M = (\mathcal{D}, \mathcal{I})$.

$(\exists x: Qx)Px$ is true in $M \therefore \bar{Q} \cap \bar{P}$ ^{buc} infinite $\bar{Q} - \bar{P}$ ^{evf} finite.

and $(\exists x: Rx)Qx$ is true in M

$\bar{R} \cap \bar{Q}$ ^{cuf} infinite $\bar{R} - \bar{Q}$ ^{dug} finite.

but $(\exists x: Rx)Px$ is false in M (1) $\bar{R} \cap \bar{P}$ ^{cud} finite OR (2) $\bar{R} - \bar{P}$ ^{fug} infinite.



Case (1): $c \cup d$ finite $\therefore c, d$ finite.

$\therefore f$ is infinite (since $c \cup f$ is infinite)

$\therefore e \cup f$ is infinite $\times$ $e \cup f$ finite.

Case (2): $f \cup g \subseteq \frac{(e \cup f) \cup (d \cup g)}{\uparrow \text{finite}} \quad \times$

$\uparrow$
infinite

4. a.

Suppose (for ~~x~~) φ is NC-valid but not SC-valid.

$\therefore \varphi$ evaluates 0 in a world of an SC-model $\mathcal{M}$

but $\mathcal{M}$ is also an NC-model ~~x~~ φ is NC-valid.

4.b.: The sentence is SC valid. $[LHS \Rightarrow RHS]$

Suppose at world w in SC-model $(W, \leq, I)$

the LHS is true and RHS is false.

then $V(\sim(P \Box \rightarrow \sim Q), w) = 1 \therefore V(P \Box \rightarrow \sim Q, w) = 0$

$\therefore$ there is a max. chr. P world to w , w_P

and $V(\sim Q, w_P) = 0 \therefore V(Q, w_P) = 1$

$\therefore V(P \Box \rightarrow Q, w) = 1$

(by anti-symmetry, w_P is unique max. chr. P world to w)

also $V(P, w_P) = 1$

$\therefore V(\Box P, w) = 1$

$\therefore$ RHS is true at w ~~X~~

$[RHS \Rightarrow LHS]$.

Suppose, for w and $(W, \leq, \mathbb{I})$, as before,

RHS is true and LHS is false.

$V(\Diamond P, w) = 1 \therefore V(P, u) = 1$ for some $u \in W$
($\max$ ^{down} P -world to w).

$\therefore$ by Limit, there is w_p

$V(P \Box \rightarrow Q, w) = 1 \therefore V(Q, w_p) = 1 \therefore V(\sim Q, w_p) = 0$
 $\therefore V(P \Box \rightarrow \sim Q, w) = 0$

$\therefore V(\sim(P \Box \rightarrow \sim Q), w) = 1$

$\therefore$ LHS is false at w .

b.i. The sentence is not NC-valid

$$w: \quad \overset{1}{\sim} (\overset{0}{P} \overset{0}{\supset} \overset{0}{\sim} Q) \overset{0}{\leftrightarrow} (\overset{0}{\Diamond} P \overset{0}{\wedge} P \overset{0}{\supset} Q)$$

$$a: \quad \overset{1}{P} \overset{0}{\sim} \overset{1}{Q}$$

$$b: \quad \overset{1}{P} \overset{0}{Q}$$

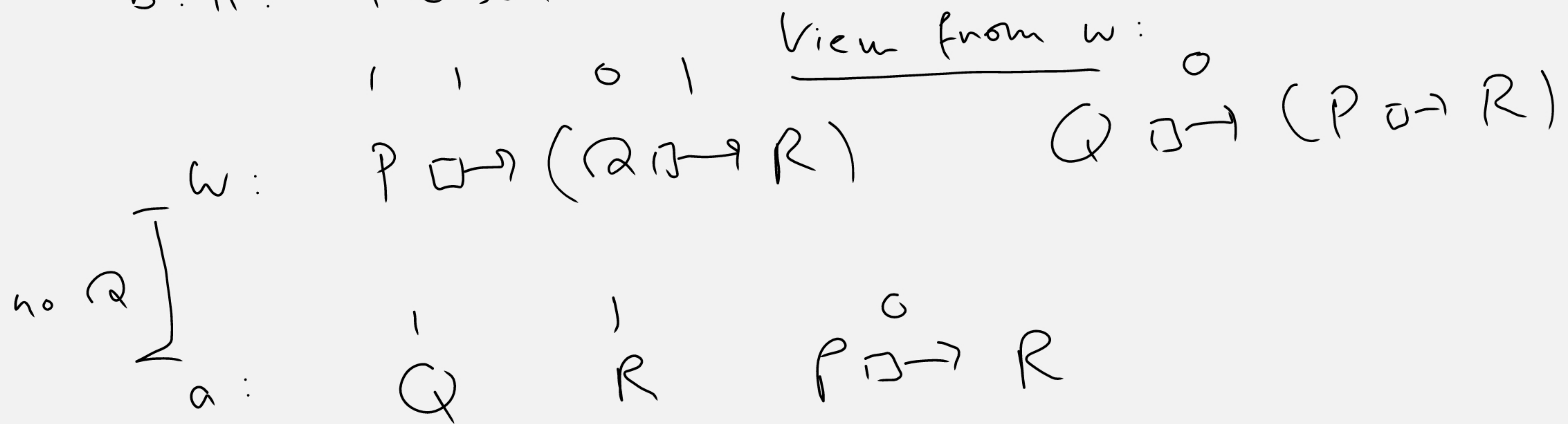
official countermodel.

$$W = \{w, a, b\}$$

$$\leq_w = \{a \leq_w b, b \leq_w a, \dots\}$$

$$I(P, a) = I(P, b) = I(Q, a) = 1 \quad (\text{otherwise } 0)$$

b. ii. The sentence is not SC-valid (and also not NC-valid, by (a))



official countermodel.

$$W = \{w, a\}$$

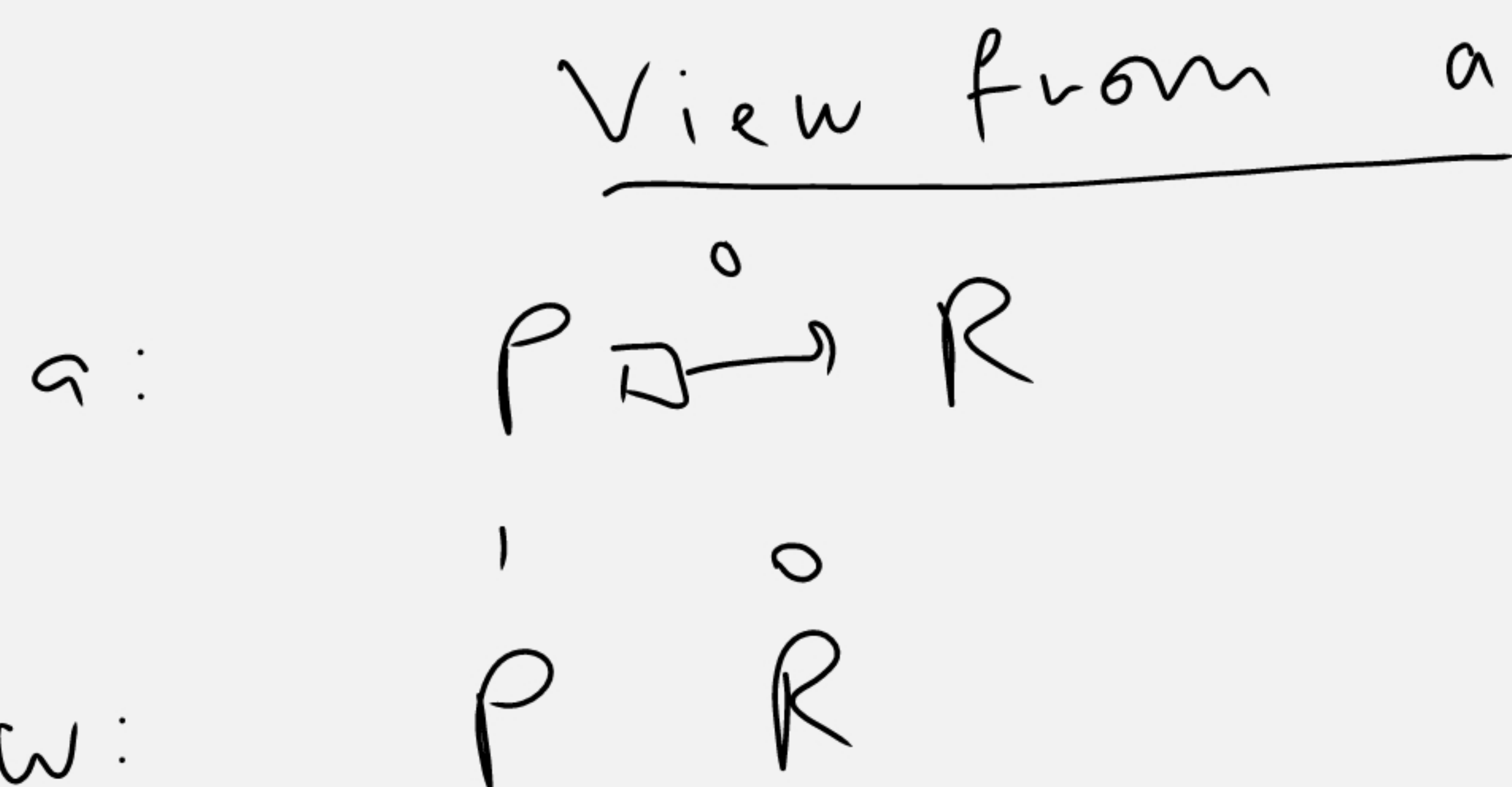
$$\leq_w = \{ \dots \}$$

$$\leq_a = \{ \dots \}$$

$$I(P, w) = I(Q, a)$$

$$= I(R, a) = 1$$

(all 0 otherwise).



So, i. β is substitute for α

as a $\varphi(P)$ with for $\varphi(\alpha)$ by replacing

terms or more free occurrences of α

with β (the name of β).

a. ii. STP, the following hold

Assuming β is substit for α .

$$[\alpha]_g = [\beta]_g \Rightarrow \forall_g (\varphi(\alpha), w) = \forall_g (\varphi(\beta), w)$$

pf by induction on φ .

$$\varphi(\beta) = \prod \beta_1 \dots \beta_n$$

Base case $\varphi(\alpha) = \prod \alpha_1 \dots \alpha_n$,

$$\beta_i = \alpha_i \text{ if } \alpha_i \neq \alpha$$

$$\beta_i = \alpha \text{ or } \beta \text{ if } \alpha_i = \alpha$$

Suppose $[\alpha]_g = [\beta]_g$. Then $[\alpha_i]_g = [\beta_i]_g$

(by defn of β_i)

$$\therefore \forall_g (\varphi(\alpha), w) = 1 \text{ iff}$$

$$\text{iff}$$

$$\langle [\alpha_1]_g, \dots, [\alpha_n]_g \rangle \in I_w(\prod)$$

$$\langle [\beta_1]_g, \dots, [\beta_n]_g \rangle \in I_w(\prod)$$

$$\text{iff } \forall_g (\varphi(\beta), w) = 1$$

• $\varphi(\alpha) = \alpha_1 = \alpha_2$. Similarly.

Suppose claim holds for formula less complex than $\varphi(\alpha)$.

• $\varphi(\alpha) = \Box \psi(\alpha)$, $\varphi(\beta) = \Box \psi(\beta)$. Suppose $[\alpha]_g = [\beta]_g$.

$V_g(\varphi(\alpha), w) = 1$ iff $V_g(\psi(\alpha), u) = 1 \quad \forall u \in w$

iff
(IH) $V_g(\psi(\beta), u) = 1 \quad \forall u \in w$

iff $V_g(\varphi(\beta), w) = 1$.

• $\sim$, $\rightarrow$ by similar argument.

$$5b.i. \quad x=y \rightarrow (\Box x=x \rightarrow \Box x=y) \quad II$$

One line proof!

$$b.ii. \quad x=x \quad R_x$$

$$\Box x=x \quad Nec.$$

$$x=y \rightarrow (\Box x=x \rightarrow \Box x=y) \quad II$$

$$x=y \rightarrow \Box x=y \quad PL.$$

$$b.iii. \quad \sim \Box x=y \rightarrow \sim x=y \quad b.ii, PL$$

$$\Box \sim \Box x=y \rightarrow \Box \sim x=y$$

$$\sim x=y \rightarrow \Box \sim \Box x=y$$

$$\sim x=y \rightarrow \Box \sim x=y$$

NEC, K, MP

Lemma 1.

PL

Lemma 1: $\sim \varphi \rightarrow \Box \sim \Box \varphi$

pf: $\Diamond \Box \varphi \rightarrow \Box \varphi \quad SS$

$$\Box \varphi \rightarrow \varphi \quad T$$

$$\Diamond \Box \varphi \rightarrow \varphi \quad PL$$

$$\sim \varphi \rightarrow \sim \Diamond \Box \varphi \quad PL$$

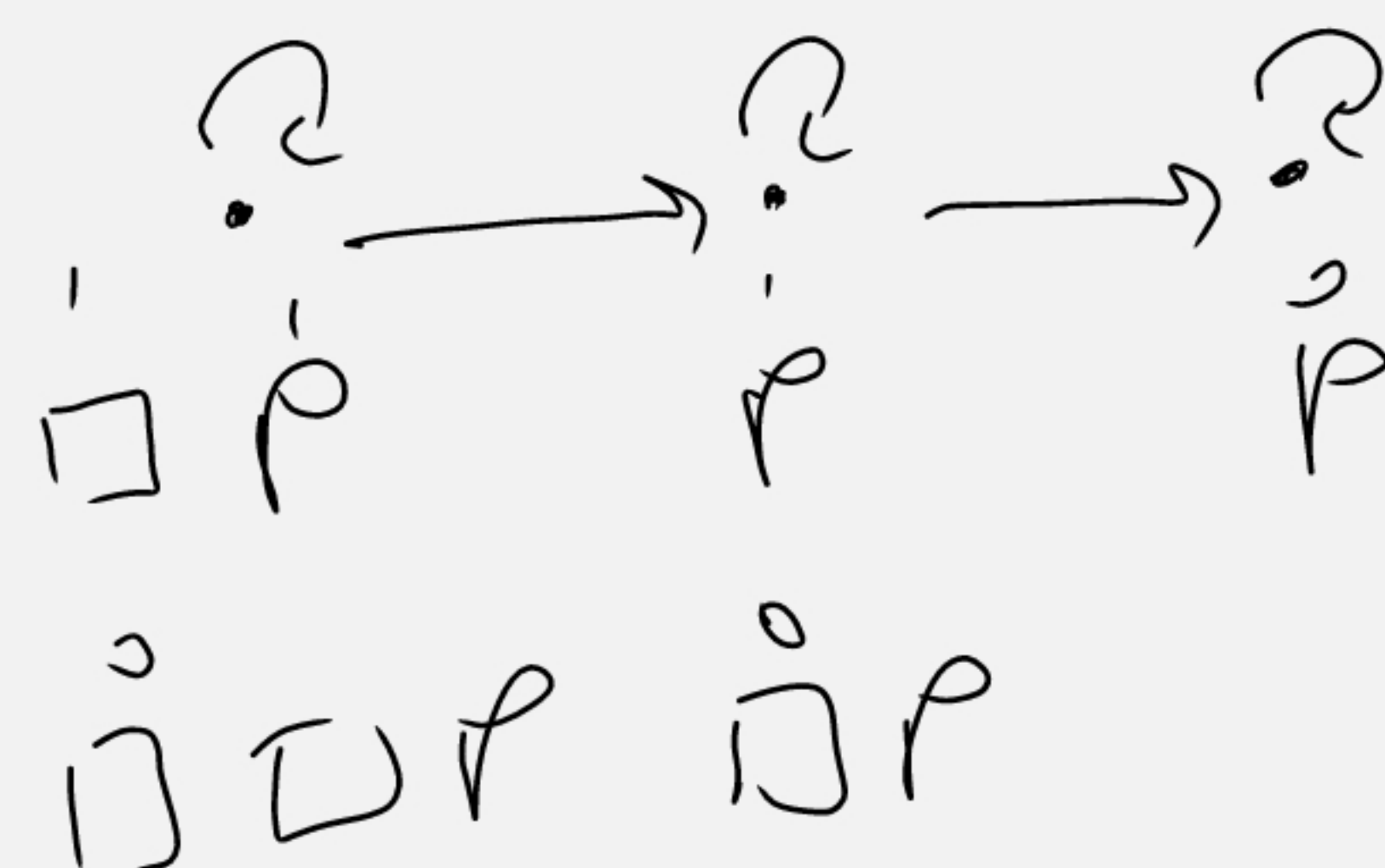
i.e. $\sim \varphi \rightarrow \sim \sim \Box \sim \Box \varphi$

$$\sim \varphi \rightarrow \Box \sim \Box \varphi \quad PL$$

$\neg \neg 22$

1. a. i. $\not\models_{\neg} \Box \Box \varphi \leftrightarrow \Box \varphi$

official countermodel.



$$W = \{u, v, w\}$$

$$R = \{ \langle u, u \rangle, \langle v, v \rangle, \langle w, w \rangle, \langle u, v \rangle, \langle v, w \rangle \}$$

$$I(p, u) = I(p, v) = 1 \quad \text{but otherwise} \quad I(x, x) = 0$$

$$a.i. \quad \models \Box \Diamond \Box \varphi \leftarrow \Box \varphi$$

SS

Suppose (for $\neg$) the formula is false at w in SS-model $\langle W, R, I \rangle$

Case 1: $V(\Box \Diamond \Box \varphi, w) = 1$ (*)

$V(\Box \varphi, w) = 0$

$\therefore V(\varphi, u) = 0$ for u s.t. Rwu (†)

but (*) : $V(\Diamond \Box \varphi, u) = 1$

$\therefore \exists v$ s.t. Ruv s.t. $V(\Box \varphi, v) = 1$

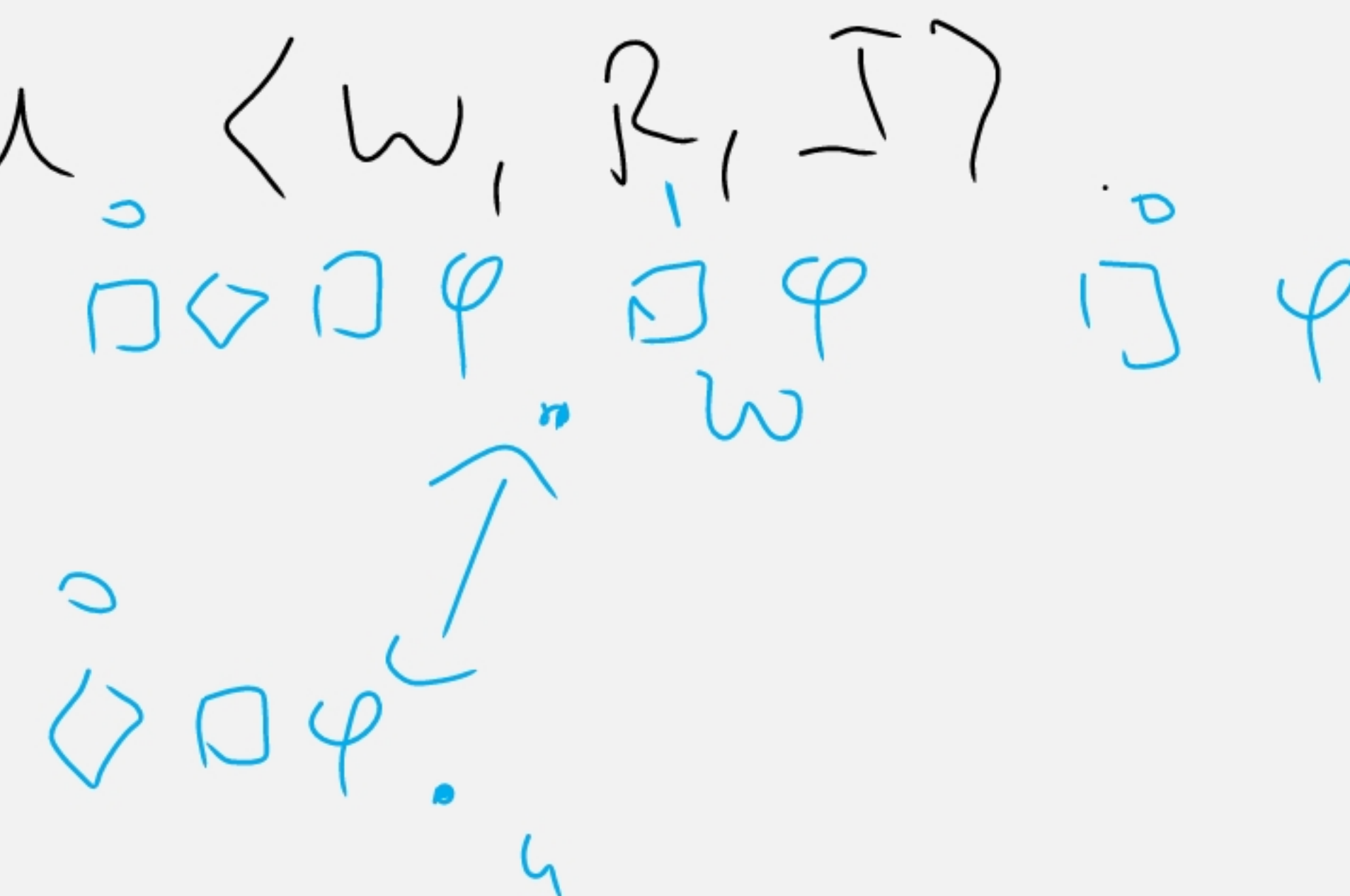
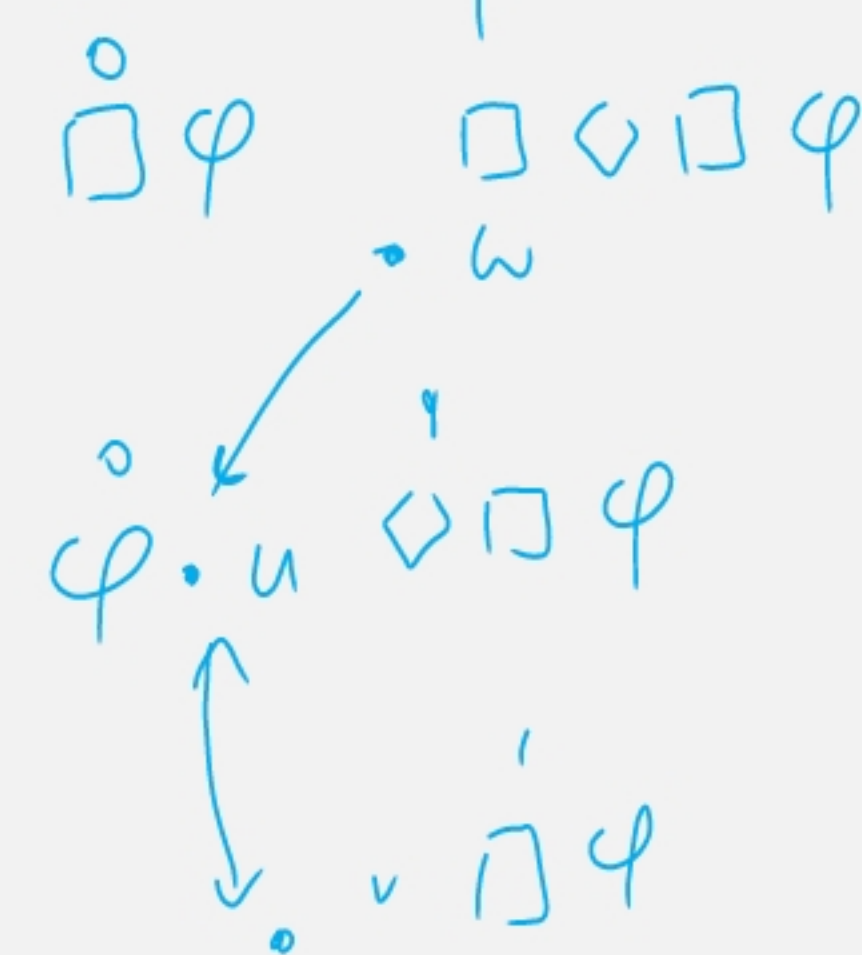
By symmetry of R , Rvu : $V(\varphi, u) = 1$ ~~(†)~~ (†)

Case 2: $V(\Box \varphi, w) = 1$ (**)

$V(\Box \Diamond \Box \varphi, w) = 0$

$\therefore \exists u$ s.t. Rwu $V(\Diamond \Box \varphi, u) = 0$

By symmetry, Ruw : $V(\Box \varphi, w) = 0$
~~(†)~~ (**)



a. iii

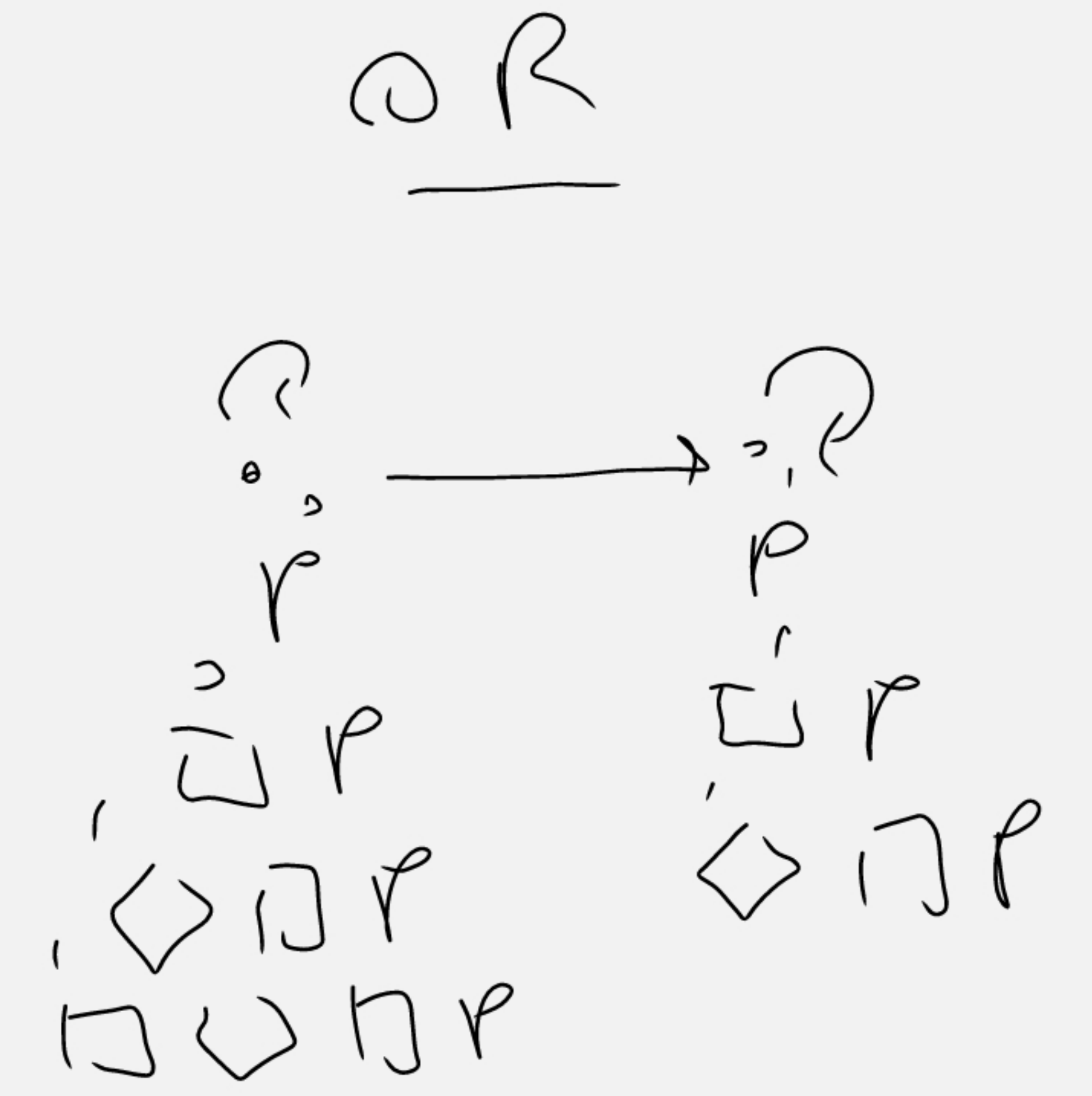
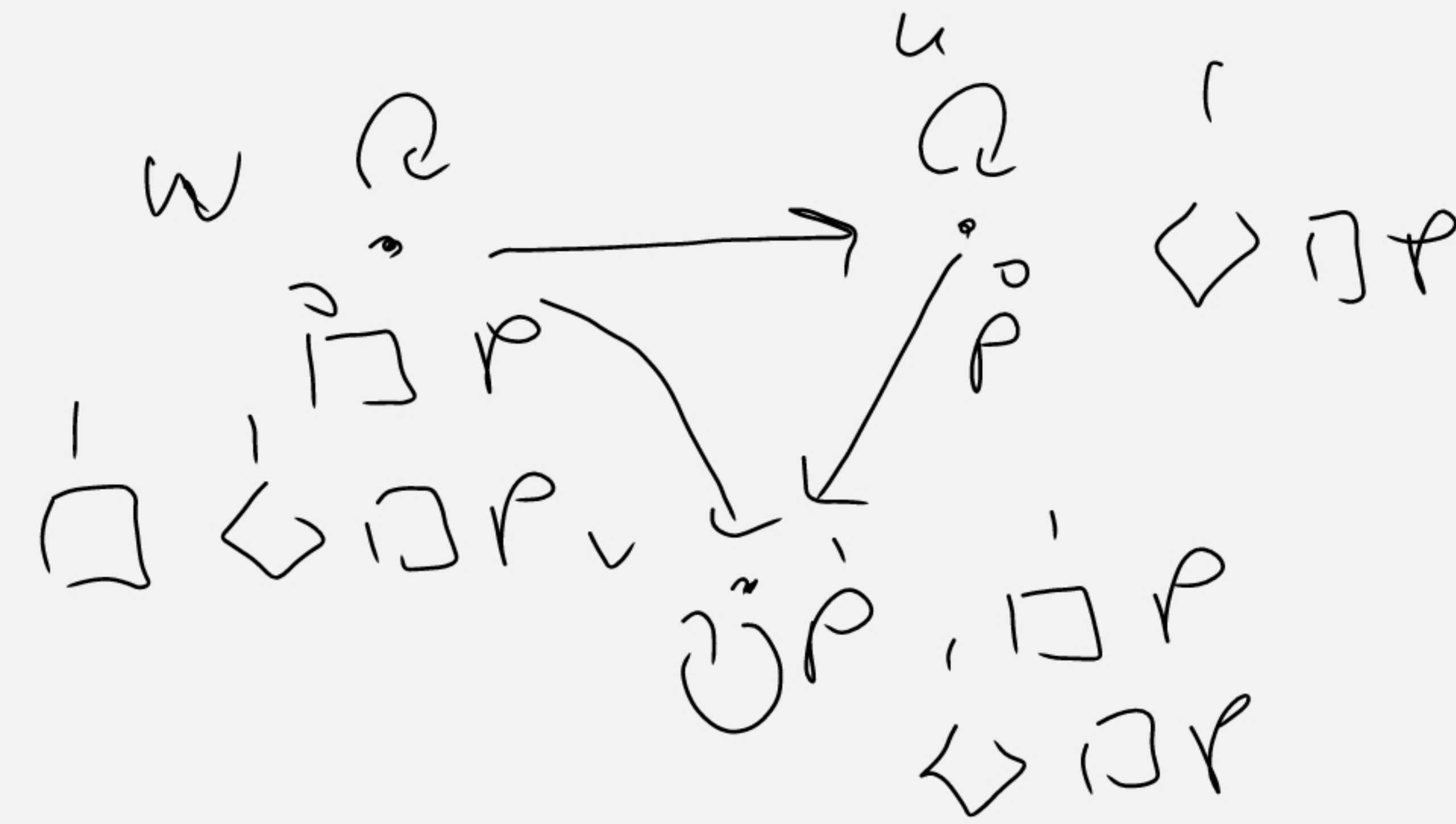
$$\models_{S_4} \Box \Diamond \Box \varphi \leftrightarrow \Box \varphi$$

Official countermodel

$$W = \{w, u, v\}$$

$$R = \{ \langle w, u \rangle, \langle u, v \rangle, \langle w, v \rangle, \langle w, w \rangle, \langle u, u \rangle, \langle v, v \rangle \}$$

$$I(p, v) = 1 \quad (\text{otherwise } I(x, x) = 0)$$



b.i. Pf by induction on n .

Base case, $n = 0$.

Immediate since $\Box^0 \varphi \rightarrow \Box^0 \psi$
is just $\varphi \rightarrow \psi$

I.H. Suppose the claim holds for n .

Consider $n+1$.

Suppose $\models_K \varphi \rightarrow \psi$ Lemma
 $\vdash \Box \varphi \rightarrow \Box \psi$
 $\vdots$
 $\vdash \Box^n \Box \varphi \rightarrow \Box^n \Box \psi$
 I.H. $\vdash \Box^n \Box \varphi \rightarrow \Box^n \Box \psi$
 a) we get.

Lemma: if $\models_K \varphi \rightarrow \psi$, then $\models_K \Box \varphi \rightarrow \Box \psi$

Suppose $\models_K \varphi \rightarrow \psi$ but $\not\models_K \Box \varphi \rightarrow \Box \psi$ (for $\exists$)

$\therefore$ we have a w in K -model $\langle w, R, I \rangle$

s.t. $V(\Box \varphi \rightarrow \Box \psi, w) = 0$

$\therefore V(\Box \varphi, w) = 1$

and $V(\Box \psi, w) = 0$. (†)

$\therefore \exists u$ s.t. Rwu and $V(\varphi, u) = 1$

but (by †) $V(\psi, u) = 0$

$\therefore V(\varphi \rightarrow \psi, u) = 0$

$\therefore \not\models_K \varphi \rightarrow \psi$.

$$\text{b.i.} \quad \models_B \Box^n \Box^n \varphi \rightarrow \varphi$$

PF by induction on n .

Base case: immediate

$$\text{since } \Box^0 \Box^0 \varphi = \varphi$$

$$\text{IH: Suppose } \models_B \Box^n \Box^n \varphi \rightarrow \varphi$$

$$\text{Consider } n+1. \quad \text{RTP: } \models_B \Box^n \Box \Box \Box^n \varphi \rightarrow \varphi$$

$$\text{STR: by IH } \models_B \Box^n \Box \Box \Box^n \varphi \rightarrow \Box^n \Box^n \varphi$$

$$\text{STR: by b.i.* } \models_B \Box \Box \Box^n \varphi \rightarrow \Box^n \varphi. \quad (\text{the same argument works in } \Box)$$

$$\text{STR } \models_B \Box \Box \varphi \rightarrow \varphi \quad [\text{insert semantic argument}].$$

$$\varphi(\alpha) = \forall \alpha \varphi(\alpha)$$

$$\varphi(\beta) = \varphi(\alpha) \quad \text{So claim holds.}$$

$$\varphi(\alpha) = \forall \beta \varphi(\alpha)$$

since β is substit for α , $\varphi(\alpha)$ has no free α .

$$\varphi(\beta) = \varphi(\alpha) \quad \text{So claim holds}$$

$$\varphi(\alpha) = \forall \gamma \varphi(\alpha), \quad \gamma \neq \alpha, \beta$$

$$\text{Suppose } [\alpha]_g = [\beta]_g \therefore [\alpha]_{g_u^x} = [\alpha]_g = [\beta]_g = [\beta]_{g_u^x} \quad (*)$$

$$V_g(\varphi(\alpha), w) = 1 \text{ iff } \forall u \in D \quad V_{g_u^x}(\varphi(\alpha), u) = 1$$

(by $(*)$) iff
(IH)

$$\forall u \in D \quad V_{g_u^x}(\varphi(\beta), u) = 1$$

$$\text{iff } V_g(\varphi(\beta), w) = 1$$