Caesar and Stipulation

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I. Introduction

II. The Caesar problem

III. A stipulative solution – two motivations

IV. Two objections

The Caesar problem

Might HP define number terms? Frege objects:

> ...our proposed definition ...does not provide for all cases. It will not, for instance, decide for us whether [Julius Caesar] is the same as [the number of Xs]...Naturally no one is going to confuse [Caesar] with [the number of Xs]; but that is no thanks to our definition of [number]. That says nothing as to whether the proposition ['$\#X = q$'] should be affirmed or denied, except for the one case where q is given in the form of [$\#Y$]. (*Grundlagen*, §66)

- HP stipulates content for unmixed contexts, i.e. '$\#X = \#Y$'
- 'says nothing' about mixed contexts, e.g.

$$\#X = \text{Caesar} \qquad\qquad \#X = \dagger Y$$

(or other atomic contexts, e.g. '$\#X$ is Roman')

A stipulative solution?

Problem needs unpicking – but if 'Caesar questions' need deciding, why not supplement HP?

Grundgesetze: are truth-values value-ranges?

– Frege stipulates, in effect, T = {T} and F = {F}

Grundlagen: are numbers Romans? are directions nations?

– Dummett (1978): 'direct stipulation' – 'straightforward'

– piecemeal stipulation – not hugely popular:

> 'Plainly, Frege is not here offering a solution to the Caesar problem: A piecemeal 'solution' is not a solution to the problem but a recipe for side-stepping it.' (Heck 2005, n. 17)

(rare exception: Linnebo, 2018)

Objection #1 | wrong answers

Macbride – stipulation may conflict with ‘antecedent facts’:

> Suppose that Caesar leads a double life. Suppose that in addition to leading his material existence Caesar is also a number. In that case the stipulation that sentences that say Caesar is a number are all false cannot succeed. For some of these sentences will be true and true sentences cannot be stipulated to be false. . . . Stipulation cannot suffice as a basis for determining that Caesar is no number. (2006, 192)

Objection #2 | incoherence

Hale & Wright – piecemeal stipulations risk incoherence:

Grundgesetze: stipulate $a = \{a\}$ – incoherent

> ...before we can safely stipulate that some object ...is a certain extension, we need an assurance that it is not (behind our back, as it were) some other extension—else our new stipulation might conflict with the original stipulation of identity-conditions ...A solution to the Caesar Problem is thus presupposed, and cannot be provided, by generalizing the kind of stipulation Frege envisages for truth-values. (2001, n. 8)

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Abstraction – a metasemantic sketch

To get clear on the problem:

how abstraction works – standard version

- phase 1: if need be, add term-forming operators (e.g. #)
- phase 2: stipulate sentential contents for unmixed contexts (e.g. #X = #Y)
- phase 3: subsentential semantic values selected that compositionally determine the stipulated sentential content.

– phase 2: abstraction principle – 'unmixed postulate':

$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}_\sigma$, tfae: $\quad \sigma \boldsymbol{x} = \sigma \boldsymbol{y}; \quad \boldsymbol{x} \sim_{\sigma:\sigma} \boldsymbol{y}$

- values of $\boldsymbol{x}, \boldsymbol{y}$: 'specifications'
- $\sim_{\sigma:\sigma}$: 'unity relation'

Caesar problem

Caesar – 'more heads than the hydra' (Heck 2016, n. 12)

– semantic aspect: HP-abstraction – **inconsistent triad**:

C1 the attempt determines a unique referent for $\#X$
(and leaves the referent of 'Caesar' unchanged)

C2 confers standard syntax/semantics on identity predicate

C3 settles no determinate truth-value for '$\#X$ = Caesar'

– solution: well-motivated rejection of **C1**, **C2**, or **C3**

Two solutions – set aside

radical indeterminacy (cf. Boccuni and Woods 2020)

- reference of '$\#X$' – radically indeterminate
- mixed contexts lack determinate truth-values

category mistake (cf. Heck 1997)

- mixed contexts – syntactically or semantically defective

– fully general, across-the-board versions overgenerate:

some Caesar questions need answers:

(1) Is $\#X = 0_{\mathbb{N}}$? (2) Is $\#X \in \mathbb{N}$? (3) Is $\#X$ non-concrete?

(1)–(3) need answers:

- to explain how *the natural numbers* are given to us
- to sustain a broadly platonist metaphysics

Two more solutions

additional desideratum – settle some Caesar questions

wholesale extraction: (Hale & Wright 2001, Rosen & Yablo 2020)

- content of mixed contexts 'extractable from' content of unmixed contexts
- 'latent content' in HP/background metaphysics
  - H&W: 'criterion of identity' for 'pure sortal'/categories
  - R&Y: 'real definition'/ essentialist metaphysics
- semantic value of # determined by HP alone

piecemeal stipulation: (Linnebo 2018, Studd 2023)

- mixed contexts open to stipulation
- semantic value of # determined by HP + other stipulations
- indeterminacy – reduced with additional stipulations

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Motivation #1 | why not?

– natural generalization of standard story:

how abstraction works – piecemeal version

- phase 1: if need be, add term-forming operators
- phase 2: stipulate sentential contents for unmixed [and mixed] contexts [or other atomic contexts]
- phase 3: subsentential semantic values selected that compositionally determine the stipulated sentential content.

Consider HP and 'Upper Hume' (Cook 2009):

$\forall \boldsymbol{X}, \boldsymbol{Y}$, tfae: $\hat{\#}\boldsymbol{X} = \hat{\#}\boldsymbol{Y}$; $\boldsymbol{X}$ and $\boldsymbol{Y}$ are equinumerous

– why not also stipulate the following?

$\forall X, Y$, tfae: $\#X \leq \#Y$; there is an injection $X \to Y$

$\forall X, \boldsymbol{Y}$, tfae: $\#X \leq \hat{\#}\boldsymbol{Y}$; there is an injection $X \to \boldsymbol{Y}$

– why not also mixed identity contexts? (cf. Heck 1997)

$\forall X, \boldsymbol{Y}$, tfae: $\#X = \hat{\#}\boldsymbol{Y}$; X and $\boldsymbol{Y}$ are equinumerous

$\forall X, \forall n \in \mathbb{N}$, tfae: $\#X = n$; X and $\{1, \ldots, n\}$ are equinumerous

– in general, if we stipulate unmixed postulates:

$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}_\sigma$, tfae: $\quad \sigma \boldsymbol{x} = \sigma \boldsymbol{y}; \quad \boldsymbol{x} \sim_{\sigma:\sigma} \boldsymbol{y}$

– perhaps also, e.g.: $\mathscr{I}_\sigma^R$ – ‘instantiation relation’

$\forall \boldsymbol{x} \in \mathcal{D}_\sigma$, tfae: $\quad R(\sigma \boldsymbol{x}); \quad \mathscr{I}_\sigma^R(\boldsymbol{x})$

– why not also ‘mixed postulates’?

$\forall \boldsymbol{x} \in \mathcal{D}_\sigma \forall \boldsymbol{y} \in \mathcal{D}_\rho$, tfae: $\quad \sigma \boldsymbol{x} = \rho \boldsymbol{y}; \quad \boldsymbol{x} \sim_{\sigma:\rho} \boldsymbol{y}$

$\forall \boldsymbol{x} \in \mathcal{D}_\sigma \forall q \in \mathcal{D}_q$, tfae: $\quad \sigma \boldsymbol{x} = q; \quad \boldsymbol{x} \sim_{\sigma:q} q$

– e.g., for Caesar:

For any X and Roman q, tfae: $\quad \#X = q; \quad \bot$

Motivation #2 | more freedom

Parable – imagine a community patch up BLV:

$\forall X, Y$, tfae: $\{X\} = \{Y\}$; X and Y coextensive or both BIG

$\forall X, x$, tfae: $x \in \{X\}$; X small and Xx

– set: $\{X\}$ for small X (suitable 'BIG'; small := non-BIG)

– familiar issue: sets lack absolute complements

Response: more abstracts! (cf. e.g. Forster 2008)

$\forall X, Y$, tfae: $\{X\}^{\complement} = \{Y\}^{\complement}$; X and Y coextensive or both BIG

$\forall X, x$, tfae: $x \in \{X\}^{\complement}$; X small and $\neg Xx$

$\forall X, Y$, tfae: $\{X\} = \{Y\}^{\complement}$; $\perp$

– complemented or c-set: $\{X\}$ or $\{X\}^{\complement}$ for small X

Prop. The c-sets make up a Boolean algebra:

$$\begin{array}{rcc} 0 := \{\Lambda\} & \{X\} \vee \{Y\} := \{X \cup Y\} & \{X\} \wedge \{Y\} := \{X \cap Y\} \\ 1 := \{\Lambda\}^{\complement} & \{X\}^{\complement} \vee \{Y\}^{\complement} := \{X \cap Y\}^{\complement} & \{X\}^{\complement} \wedge \{Y\}^{\complement} := \{X \cup Y\}^{\complement} \\ \neg\{X\} := \{X\}^{\complement} & \{X\} \vee \{Y\}^{\complement} := \{X^c \cap Y\}^{\complement} & \{X\} \wedge \{Y\}^{\complement} := \{X \cap Y^c\} \\ \neg\{X\}^{\complement} := \{X\} & \{X\}^{\complement} \vee \{Y\} := \{X \cap Y^c\}^{\complement} & \{X\}^{\complement} \wedge \{Y\} := \{X^c \cap Y\} \end{array}$$

$\Lambda := \lambda x.x \neq x;\quad X^c := \lambda x.\neg Xx;\quad X \cup Y := \lambda x(Xx \vee Yx),\quad$ etc.

– **but:** can we thus introduce c-sets?
  - piecemeal stipulation: yes (given suitable 'BIG')
  - wholesale extraction: no (or so I will argue)

– **moral:** wholesale extraction curtails mathematical freedom

– why does wholesale extraction curtail freedom?

NV If σ- and ρ-abstracts introduced by notational variants of same abstraction principle, σ and ρ have same semantic value:

$$\text{therefore, for any } \boldsymbol{x} \in \mathcal{D}_\sigma,\ \sigma\boldsymbol{x} = \rho\boldsymbol{x}$$

– **Wholesale:** endorse **NV**

- meaning of σ – determined just by the unity relation, $\sim_{\sigma:\sigma}$
- by **NV** $\{X\} = \{X\}^{\complement}$
- not free to introduce csets as above. e.g.:

$$\emptyset = \emptyset^{\complement} \qquad \emptyset \notin \emptyset \qquad \emptyset \in \emptyset^{\complement}$$

– **Piecemeal:** reject **NV**

- meaning of σ – not just determined by unmixed postulates
- restore coherence – free to deny $\{X\} = \{X\}^{\complement}$

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Objection #1 | wrong answers

MacBride: might we stipulate the wrong answer?

For any X and any Roman q, tfae:

$\#X = q$; q is a dictator of the Roman Republic and the class of dictators succeeding q is equinumerous with X

Or again consider Shapiro's CP alongside HP.

$\forall X, Y \subseteq \mathbb{Q}$, tfae:

$\sup X = \sup Y$; X and Y have same rational upper bounds

– Community 1 identify their #- and sup-abstracts:

$\forall X, \forall Y \subseteq \mathbb{Q}$, tfae:

$\#X = \sup Y$; Y has same rational upper bounds as $\{0_{\mathbb{Q}}, \ldots, n_{\mathbb{Q}}\}$, and X is equinumerous with $\{0_{\mathbb{Q}}, \ldots, n_{\mathbb{Q}}\} \setminus \{0_{\mathbb{Q}}\}$.

– Community 2 distinguish theirs:

$\forall X, \forall Y \subseteq \mathbb{Q}$, tfae $\#X = \sup Y$; $\bot$

– can both be right?

– **reply:** sort of – depends what you mean by 'right'

To clarify – consider an ‘unmixed’ case:

– Community 1 lay down HP:

$\forall X, Y$, tfae: $\#X = \#Y$; X and Y are equinumerous

– Community 2 take a pre-Cantorian stance:

$\forall X, Y$, tfae: $\#X = \#Y$; X and Y are equinumerous or both infinite

– can both be right?

Success: do both abstraction attempts succeed (individually)?
– yes, both introduce cardinal-like abstracts

Reduction: are these abstracts the familiar cardinals?

- the stipulations accord different referents to #: $\#^1$ and $\#^2$
- at most one is $\#^*$, the ‘intended’ cardinality-operator:

$$\#^* X := \text{the cardinality of } X$$

Similar considerations apply in 'mixed cases':

- Community 1, recall, 'identify' their #- and sup-abstracts
- Community 2 distinguish theirs

Success: do the abstraction attempts succeed?
– yes, both introduce cardinal-*like* and real-*like* abstracts

Reduction: are these abstracts the familiar cardinals and reals?

- as before, stipulations introduce $\#^1/\#^2$ and $\mathbf{sup}^1/\mathbf{sup}^2$
- in at most one case, $\#^i = \#^*$ and $\mathbf{sup}^i = \mathbf{sup}^*$ (i = 1 or 2)

Moral: reduction, not success, hostage to ‘antecedent’ facts:

For any X and Roman q, tfae: $\#X = q$; $\bot$

- Caesar leads a double life: may still introduce (non-Roman) cardinal-*like* abstracts
- sane case: combined with other mixed postulates – may yet suffice to pick out $\#^*$

Objection #2 | incoherence

Hale and Wright: piecemeal stipulation risks incoherence

Reply:

- abstraction risks incoherence: bad company
- response: seek success criterion

(focus: my favourite response to bad company)

– piecemeal abstraction – patchwork of unity relations:

$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}_\sigma$, tfae: $\sigma \boldsymbol{x} = \sigma \boldsymbol{y}$; $\boldsymbol{x} \sim_{\sigma:\sigma} \boldsymbol{y}$

$\forall \boldsymbol{x} \in \mathcal{D}_\sigma \forall \boldsymbol{y} \in \mathcal{D}_\rho$, tfae: $\sigma \boldsymbol{x} = \rho \boldsymbol{y}$; $\boldsymbol{x} \sim_{\sigma:\rho} \boldsymbol{y}$

$\forall \boldsymbol{x} \in \mathcal{D}_\sigma \forall q \in \mathcal{D}_q$, tfae: $\sigma \boldsymbol{x} = q$; $\boldsymbol{x} \sim_{\sigma:q} q$

$\forall \boldsymbol{x} \in \mathcal{D}_\sigma$, tfae: $R(\sigma \boldsymbol{x})$; $\mathscr{I}^R_\sigma(\boldsymbol{x})$

– necessary condition for success:

- $\sim_{\sigma:\sigma}, \sim_{\sigma:\rho}, \sim_{\sigma:q}$, induce global unity relation: $\sim$
- $\mathscr{I}^R_\sigma, \mathscr{I}^R_\rho$, etc. induce global instantiation relation: $\mathscr{I}^R$

Congruence: $\sim$ an equivalence relation, respected by each $\mathscr{I}^R$

Is **Congruence** sufficient for success?

Orthodox view: clearly not!

- BLV meets **Congruence**
- impredicative/static: abstracts in pre-abstraction domain

My preferred view: yes

- predicative/dynamic: abstracts may be '**new**'
- dynamic BLV – unproblematic
- model-theoretic safety result: if an abstraction attempt meets **Congruence**, then some interpretation extends the pre-abstraction interpretation according to its postulates