127 Philosophical Logic Exam Solutions

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Question 1

(a)(i). **Claim:** if $\vdash_{\mathrm{D}}^{k} \phi$, then $\vdash_{\mathrm{D}} d(\phi)$. We prove by induction on the length k of the proof. Here, we use the superscript $\vdash^{k} \phi$ to denote a proof of ϕ in k steps.

Base case: $\vdash_{\mathrm{D}}^{0} \phi$, i.e. ϕ is an axiom. Consider the cases:

- ϕ is an instance of PL1.
  Observe that d distributes across $\to$, so applying d to an instance of PL1 gives another instance of PL1. That is, $d\big(\psi \to (\chi \to \psi)\big) = d(\psi) \to \big(d(\chi) \to d(\psi)\big)$. So $\vdash_{\mathrm{D}} d(\phi)$.
- ϕ is an instance of PL2 or PL3.
  Same reasoning as previous case.
- ϕ is an instance of K, i.e. $\phi = \Box(\psi \to \chi) \to (\Box\psi \to \Box\chi)$.
  Then, $d(\phi) = (\psi \to \chi) \to (\psi \to \chi)$, which is a D-theorem by PL.
- ϕ is an instance of D, i.e. $\phi = \Box\psi \to \Diamond\psi$.
  Writing $\Diamond$ using $\Box$, we have $d(\phi) = \psi \to {\sim}{\sim}\psi$, which is a D-theorem by PL.

Inductive step: Take $k > 0$. IH: the claim holds for any proof less than k steps. Now suppose $\vdash_{\mathrm{D}}^{k} \phi$ and consider the last line of the proof. Consider the cases:

- ϕ is an axiom.
  This is covered by the discussions for the base case.
- ϕ is obtained by NEC.
  Then, $\phi = \Box\psi$ and $\vdash_{\mathrm{D}}^{\ell} \psi$ for some $\ell < k$. We have $d(\phi) = \psi$, so $\vdash_{\mathrm{D}} \phi$.
- ϕ is obtained by MP.
  Then, $\vdash_{\mathrm{D}}^{\ell} \psi \to \phi$ and $\vdash_{\mathrm{D}}^{\ell'} \psi$ for some $\ell, \ell' < k$. By IH, we have $\vdash_{\mathrm{D}} d(\psi) \to d(\phi)$ and $\vdash_{\mathrm{D}} d(\psi)$. So $\vdash_{\mathrm{D}} d(\phi)$ by MP.

(ii). Suppose $\vdash_{\mathrm{D}} \Box\phi$. Then, by part (i), $\vdash_{\mathrm{D}} d(\Box\phi)$, i.e. $\vdash_{\mathrm{D}} \phi$.

(iii). **Claim:** $\vdash_{\mathrm{D}} \Box\phi$ iff $\vdash_{\mathrm{D}} \Diamond\phi$.

($\Rightarrow$) Suppose $\vdash_{\mathrm{D}} \Box\phi$. Then $\vdash_{\mathrm{D}} \Diamond\phi$ by D and MP.

($\Leftarrow$) Suppose $\vdash_{\mathrm{D}} \Diamond\phi$. By part (a)(i), $\vdash_{\mathrm{D}} d(\Diamond\phi)$. Note $d(\Diamond\phi) = d({\sim}\Box{\sim}\phi) = {\sim}{\sim}\phi$, so $\vdash_{\mathrm{D}} \phi$ by PL and $\vdash_{\mathrm{D}} \Box\phi$ by NEC.

(b)(i). Yes. Suppose $\vdash_{\text{KB}} \Box\phi$, then $\vdash_{\text{KB}} \phi$ by T and MP.

(ii). **Claim:** $\nvdash_{\text{KB}} \Box\Box\phi \to \phi$.

We prove this via a semantic countermodel, taking ϕ to be P.

$$\mathscr{W} = \{w\}, \quad \mathscr{R} = \varnothing$$
$$\mathscr{I}(\alpha) = 0 \text{ for every sentence letter } \alpha$$

In this countermodel, $V(\Box\Box P \to P, w) = 0$, so we conclude that $\nvDash \Box\Box P \to P$. But note that $\mathscr{R}$ is (vacuously) symmetric, and since each KB-theorem is valid in models with a symmetric accessibility relation, this means $\nvdash_{\text{KB}} \Box\Box P \to P$.

Claim: $\vdash_{\text{KB}} \Box(\Box\Box\phi \to \phi)$.

We need to somehow obtain two consecutive boxes. Consider an instance of B:

$$\begin{array}{ll} \vdash_{\text{KB}} \Diamond\Box\Box\phi \to \Box\phi & \text{B} \\ \vdash_{\text{KB}} {\sim}\Box{\sim}\Box\Box\phi \to \Box\phi & \\ \vdash_{\text{KB}} \underbrace{\Box{\sim}\Box\Box\phi}_{A} \lor \underbrace{\Box\phi}_{B} & \text{PL} \end{array}$$

The idea now is to use the following inference, where C is the desired $\Box(\Box\Box\phi \to \phi)$:

$$\frac{A \lor B \quad A \to C \quad B \to C}{C}$$

Subclaim ($A \to C$): $\vdash_{\text{KB}} \Box{\sim}\Box\Box\phi \to \Box(\Box\Box\phi \to \phi)$.

1	$\vdash_{\text{KB}} {\sim}\Box\Box\phi \to (\Box\Box\phi \to \phi)$	PL
2	$\vdash_{\text{KB}} \Box{\sim}\Box\Box\phi \to \Box(\Box\Box\phi \to \phi)$	NEC, K, MP 1

Subclaim ($B \to C$): $\vdash_{\text{KB}} \Box\phi \to \Box(\Box\Box\phi \to \phi)$.

1	$\vdash_{\text{KB}} \phi \to (\Box\Box\phi \to \phi)$	PL1
2	$\vdash_{\text{KB}} \Box\phi \to \Box(\Box\Box\phi \to \phi)$	NEC, K, MP 1

Therefore, by the PL inference, we conclude that $\vdash_{\text{KB}} \Box(\Box\Box\phi \to \phi)$, as desired. The two claims provide a counterexample for the denecessitation property of KB.

Question 2

(a)(i). Recall that an accessibility relation is <u>*serial*</u> if for any $w \in \mathscr{W}$, there is some $v \in \mathscr{W}$ with $\mathscr{R}wv$ ("every world sees at least one world").

Claim: $\Box\phi \to \Diamond\phi$ is valid on $\langle \mathscr{W}, \mathscr{R} \rangle$ iff $\mathscr{R}$ is serial on $\mathscr{W}$.

($\Rightarrow$) By contraposition. Suppose $\mathscr{R}$ is not serial on $\mathscr{W}$. Then, there exists some $w \in \mathscr{W}$ such that no $v \in \mathscr{W}$ is such that $\mathscr{R}wv$. Then, for any $\mathscr{I}$ and sentence letter P, we have that $V(\Box P, w) = 1$ but $V(\Diamond P, w) = 0$. So $V(\Box P \to \Diamond P, w) = 0$, and thus $\Box\phi \to \Diamond\phi$ is not valid on $\langle \mathscr{W}, \mathscr{R} \rangle$.

($\Leftarrow$) Suppose $\mathscr{R}$ is serial on $\mathscr{W}$. Let $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ be a model based on $\langle \mathscr{W}, \mathscr{R} \rangle$.

1 Assume for reductio that some $w \in \mathscr{W}$ is such that $V(\Box\phi \to \Diamond\phi, w) = 0$.
2 So, $V(\Box\phi, w) = 1$, and...
3 ...$V(\Diamond\phi, w) = 0$.
4 By seriality of $\mathscr{R}$, there exists some $v \in \mathscr{W}$ such that $\mathscr{R}wv$.
5 So, by (2), $V(\phi, v) = 1$.
6 But by (3), $V(\phi, v) = 0$. ※

(ii). **Claim:** $\Diamond\phi \to \Box\phi$ is valid on $\langle \mathscr{W}, \mathscr{R} \rangle$ iff $\mathscr{R}$ is a function.

($\Rightarrow$) Again by contraposition. Suppose $\mathscr{R}$ is not a function. Then, we have $w, u, u' \in \mathscr{W}$ with $\mathscr{R}wu$, $\mathscr{R}wu'$ but $u \neq u'$. Consider some sentence letter P and interpretation function $\mathscr{I}$ such that $V(P, u') = 0$ and $V(P, u) = 1$. Then, $V(\Diamond P \to \Box P, w) = 0$ in that model, and therefore $\Diamond\phi \to \Box\phi$ is not valid on $\langle \mathscr{W}, \mathscr{R} \rangle$.

($\Leftarrow$) Suppose $\mathscr{R}$ is a function. Let $\langle \mathscr{W}, \mathscr{R}, \mathscr{I} \rangle$ be a model based on $\langle \mathscr{W}, \mathscr{R} \rangle$.

1 Assume for reductio that some $w \in \mathscr{W}$ is such that $V(\Diamond\phi \to \Box\phi, w) = 0$.
2 So, $V(\Diamond\phi, w) = 1$, and...
3 ...$V(\Box\phi, w) = 0$.
4 By (2), $V(\phi, u) = 1$ for some $u \in \mathscr{W}$ with $\mathscr{R}wu$
5 By (3), $V(\phi, v) = 0$ for any $v \in \mathscr{W}$ with $\mathscr{R}wv$
6 But $\mathscr{R}$ is a function, and $\mathscr{R}wu$, $\mathscr{R}wv$, so $u = v$, and thus $V(\phi, u) = 0$. ※

(iii). **Claim:** $\Diamond\phi \leftrightarrow \Box\phi$ is valid on $\langle \mathscr{W}, \mathscr{R} \rangle$ iff R is a function with domain $\mathscr{W}$.

Note that $\mathscr{R}$ is a function with $\text{dom}(\mathscr{R}) = \mathscr{W}$

iff $\mathscr{R}$ is a function and $\mathscr{R}$ is serial on $\mathscr{W}$

iff $\Diamond\phi \to \Box\phi$ is valid on $\langle \mathscr{W}, \mathscr{R} \rangle$ and $\Box\phi \to \Diamond\phi$ is valid on $\langle \mathscr{W}, \mathscr{R} \rangle$

iff $\Diamond\phi \leftrightarrow \Box\phi$ is valid on $\langle \mathscr{W}, \mathscr{R} \rangle$

(b)(i). Suppose $\langle \mathscr{W}, \mathscr{R} \rangle$ is a universal frame.

Claim: $\Box P_1 \vee \Box(P_1 \to P_2) \vee \Box(P_1 \wedge P_2 \to P_3)$ is not valid on $\langle \mathscr{W}, \mathscr{R} \rangle$ iff $\mathscr{W}$ has 3 or more members.

($\Leftarrow$) Suppose w_1, w_2, w_3 are 3 pairwise distinct members of $\mathscr{W}$. Now consider an interpretation function $\mathscr{I}$ given by

$$\mathscr{I}(P_1, w_1) = \mathscr{I}(P_2, w_2) = \mathscr{I}(P_3, w_3) = 0, \quad \mathscr{I}(\alpha, w) = 1 \text{ otherwise}$$

Then, the formula is false at w_1 (in fact, at any w), and therefore it is not valid.

($\Rightarrow$) Suppose the formula is not valid, and therefore false in some $w \in \mathscr{W}$.

1 Then, $V(\Box P_1 \vee \Box(P_1 \to P_2) \vee \Box(P_1 \wedge P_2 \to P_3), w) = 0$
2 Then, $V(\Box P_1, w) = V(\Box(P_1 \to P_2), w) = V(\Box(P_1 \wedge P_2 \to P_3), w) = 0$
3 So, there exists $u_1 \in \mathscr{W}$ such that $V(P_1, u_1) = 0$, ...
4 ...there exists $u_2 \in \mathscr{W}$ such that $V(P_1 \to P_2, u_2) = 0$, and...
5 ...there exists $u_3 \in \mathscr{W}$ such that $V(P_1 \wedge P_2 \to P_3, u_3) = 0$
6 By (4), $V(P_1, u_2) = 1$ and $V(P_2, u_2) = 0$
7 By (5), $V(P_1, u_3) = V(P_2, u_3) = 1$ and $V(P_3, u_3) = 0$
8 By (3) and (6), $u_1 \neq u_2$
9 By (3) and (7), $u_1 \neq u_3$
10 By (6) and (7), $u_2 \neq u_3$

Therefore, $\mathscr{W}$ must have at least 3 members.

(ii). Continuing the same pattern we see in part (b)(i), let ϕ_n denote:

$$\Box P_1 \vee \Box(P_1 \to P_2) \vee \Box(P_1 \wedge P_2 \to P_3) \vee \cdots \vee \Box(P_1 \wedge \cdots \wedge P_n \to P_{n+1})$$

We can do a quick sanity check for $n = 2$ and see ϕ_2 is indeed the wff in part (b)(i).